Derivative Pricing With Strategic Competition For Liquidity

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Abstract

We consider a financial market with two large investors whose trades affect prices, so they face liquidity risk. In this setting, we examine utility based prices for derivative securities in an extended version of the canonical Black—Scholes derivative pricing model. In our model the large investors' risk preferences of are represented by an exponential utility functions. In a stylized binomial example with price impact, we show that the payoff space and the no–arbitrage pricing functional are convex but not necessarily linear, which impedes arbitrage pricing. In a continuous time framework, where large traders play a non–zero sum singular stochastic differential Cournot game, we obtain a pricing rule for derivative securities that can be characterized by a nonlinear transformation of the expectation of the distorted derivative payoff under the Markov—Nash pricing measure. Under specified assumptions, we derive a liquidity adjusted Black—Scholes equation and show that the manipulation free price coincides with the Black—Scholes price. We also implement a numerical algorithm for computing the price of European style options in a general framework.

Keywords – Asset Pricing With Liquidity Frictions, Utility Indifference Pricing, Singular Stochastic Differential Game, Liquidity Adjusted Black–Scholes Equation

JEL Classification – G13, E42, E44, L13
AMS Classification (2020) – 49N70, 91A05, 91A15, 91B54

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1. Introduction

Economists since Samuelson (1965) have stressed the importance of determining a cogent pricing rule for a derivative security - a financial asset which derives its value from that of an underlying *primary* asset. The importance of derivative securities in contemporary financial markets can be ascertained from the fact that the celebrated Black–Scholes equation for pricing European call and put options, derived in the seminal paper Black and Scholes (1973) building on foundational work of French mathematician Louis Bachelier (Bachelier, 1900), was listed by Business Insider as one of the seventeen equations that changed the course of human history (Kiersz, 2014), along with Newton's Law of gravity and Schrodinger's equation.[1]

In the decade following the pioneering contribution of Fischer Black and Myron Scholes, it was observed that while the Black–Scholes equation presented an elegant and straightforward approach to pricing of derivative securities, it relied on assumptions which often proved stringent to hold in practice (Black, 1989). In particular, the events leading to the stock market crash of 1987 called into question the assumption of perfectly liquid underlying markets, which is a cornerstone of canonical derivative pricing theory. Briefly, at the time of the crash, many institutional investors followed portfolio insurance strategies which dictated liquidation of a position if asset prices dropped below a threshold. Price triggers activated by the crash dried up available liquidity and asset prices declined significantly in response to further selling (Shiller, 1988).

Motivated by this observation, in the present work we re-examine classical derivative pricing theory in the context of illiquid financial markets, where illiquidity is caused by the presence of *Whales* or large institutional investors in the underlying asset market. Trading by these investors has an appreciable effect on the underlying price and thus the value of a derivative security written on the underlying asset. The relevance of this re-examination extends beyond traditional financial markets, as evidenced by the extreme liquidity crunch observed in cryptocurrency spot markets following unwinding of positions by large institutional investors in early December 2021, which subsequently lead to a United States House of Representatives financial services committee hearing on digital assets (Silverman, 2021).

In addition to anecdotal evidence, there is a large body of empirical literature in finance which advocates the need to look beyond the foundational works of option pricing theory that rely crucially on the assumption that investors in the financial market are small or *price takers* in the sense that an investor can buy (or sell) as much of a financial asset as they want at the market price. For example, earlier works by Glosten and Harris (1988), and Brennan and Subrahmanyam (1996) document evidence of positive risk premia associated with price impact liquidity risk. Later works, such as Sadka (2006), Korajczyk and Sadka (2008), and Sadka (2010) extend their analysis to show that price impact liquidity risk is not a historical artifact but rather a prominent feature of financial markets.

Clearly, in the light of evidence presented by these works, it is difficult to argue positively for the validity of the assumption of perfectly liquid underlying when dealing with pricing of derivatives in many asset classes of interest. Moreover, it has been argued that the optimal strategy for managing liquidity risk should account not only for own price impact but that of other large institutional investors or Whales in the market as well, see Brunnermeier and Pedersen (2005). These observations constitute the principal motivation for the present work in which we address the issue of deriving a

[1] For a comprehensive discussion of the impact of Black–Scholes equation on the functioning of financial markets refer Stewart (2012) on which the Business Insider report is based, as well as the survey article by Jarrow (1999).

well–founded notion for the price of a derivative security whose underlying primary asset is traded in an illiquid spot market by non–Walrasian investors.

To motivate our analysis, we first consider a stylized example of a two–period Binomial model with price impact, with the help of which we show that introduction of a single large investor in an otherwise standard setup leads to the failure of the replication pricing approach based on the law of one price under no arbitrage. Exploiting existing results, we show that market incompleteness caused by price impact differs crucially from incompleteness in frictionless markets, which exacerbates the limitation of replication pricing approach. As an alternative, we propose using the utility indifference price approach widely studied in the context of derivative pricing in frictionless incomplete markets, see Carr, Geman and Madan (2001), for pricing derivative securities whose underlying spot market is dominated by Whales.

The utility indifference pricing approach is rooted in the relative pricing tradition where prices of underlying traded financial assets are specified exogenously, as opposed to the absolute pricing or general equilibrium pricing framework where derivative prices are determined together with prices of primary financial assets to ensure zero net aggregate demand. Besides historical precedent, the consideration of utility indifference prices in the present work is merited by our focus on examining the pricing of derivative securities where the underlying is traded in an order–book like setup with market–order prices being determined exogenously as a function of order volume.[2]

However, relaxing the assumption of price–taking investors poses a serious technical challenge, particularly when one allows for the possibility of strategic competition for liquidity in a financial market with more than one large investor whose trading has long–term or *permanent* impact on the spot price of the primary asset. In such instances, the Hamilton associated with the optimization problem of an investor may fail to be finite valued on its domain. The resulting class of optimization problems is referred to as *singular* stochastic optimal control problems. Recall that an investor has permanent price impact on the spot price if an investor's trade at time t influences the spot price at time $\kappa > t$. The analysis of optimal portfolios and trading strategies under permanent price impact is important for practical reasons as most trading algorithms use it as a basis for their *tactical layer* responsible for generating an optimal trading schedule.

We exploit the fact that the singular optimization problem in our strategic setup resembles the singular control problem analyzed in Lasry and Lions (2000), to construct an equivalent auxiliary optimization problem which is tractable by standard methods and permits characterization of an investor's optimal portfolio of primary and derivative assets. The equivalence result between the singular optimization problem of an investor and the constructed auxiliary problem is the principal technical contribution of the present work. The novelty of the equivalence result established in this work stems from the fact that investors are allowed to hold a portfolio consisting of primary as well as derivative assets, while trading the primary asset in a market which is imperfectly competitive.

Establishing the equivalence result when investors are permitted to hold a nonzero position in the derivative asset is non–trivial, as a large investor who is cognizant of their price impact has an incentive to influence or manipulate the payoff of the derivative asset. The standard approach in existing literature on derivative pricing with price impact has been to focus on the case of non-manipulable derivative securities, that is, securities whose payoff at maturity is unaffected by a large investor's position, see Lions and Lasry (2007) and Bank and Dolinsky (2019). In contrast, we consider

[2]Interested readers should consult Carvajal (2018) for an insightful discussion of equilibrium pricing in the context of batch–auction markets.

a generalized setup where we do not assume ex-ante that the derivatives are non-manipulable, albeit at the cost of additional assumptions regarding the payoff profile of the derivative security. Nevertheless, the assumptions hold for a large class of derivatives including vanilla European call and put options as well as exotics such as the *chooser* option.

With the help of the auxiliary hedging problem and the equivalence result we are able to characterize the Markov–Nash equilibrium trading strategies and the associated indirect utility functions of the large investors as a coupled system of nonlinear partial differential Hamilton–Jacobi–Bellman equations. This system must in general be solved numerically in order to compute the indifference price of a derivative security. However, under certain assumptions on the class of permissible trading strategies, we are able to derive a semi–analytical characterization of the indifference price of a derivative in terms of a non-linear transformation of the expectation of a non-linear distortion of the derivative security payoff under the Markov–Nash equivalent martingale measure.

In this special instance, we are able to obtain a closed form expression for a liquidity adjusted version of the Black–Scholes equation, which is derived via a routine application of Feynman–Kac formula to the pricing relation obtained previously. The liquidity adjusted pricing equation we obtain contains an additional liquidity correction term, besides the standard terms in the canonical pricing equation. The liquidity correction term is a function of the delta of the derivative, that is, the sensitivity of the derivative payoff with respect to the underlying asset price. Using the rule of thumb that the delta of a European option can be proxied by the moneyness of the option, it stands to reason that the liquidity correction term we obtain is most pertinent in the case of deep in-the-money options. This finding seems to be consistent with existing empirical evidence, see for example Martin (2017), which documents that liquidity of an option is inversely related to its moneyness, thereby indicating greater liquidity risk associated with deep in-the-money options which makes the liquidity correction term more relevant for them.

We further show that the ask price of a derivative security exceeds its Black–Scholes price while the bid price is at most the Black–Scholes price, indicative of the ability of large investors to manipulate derivative payoffs and consequently the fair price of a derivative security. For the case of a European call, we present an example illustrating that the lower bound is tight in the sense that the derivative bid price can be pushed down to zero. By a simple reiteration of the auxiliary problem construction, we are able to show that the unique manipulation free price for the security coincides with the Black–Scholes price.

In order to numerically solve the coupled system of nonlinear partial differential equations characterizing the Markov–Nash equilibrium portfolios as well as compute the associated indifference price of a derivative security, we extend the algorithm proposed in Achdou, Han, Lasry, Lions and Moll (2022) based on the finite difference method. While, their algorithm does deal with a coupled system of PDE's, the system they consider comprises of a forward and a backward equation which are mutually transpose, thereby simplifying the algorithm. However, the coupled system we solve consists of two backward equations, with dynamics governed by multivariate diffusion processes which effectively requires us to first extend their algorithm from a univariate to a multivariate state variable and then reiterate the finite difference scheme to account for the strategic framework. Thus, our algorithm can be considered as a generalization of numerical schemes proposed in literature for estimating viscosity solutions of HJB equations dating back to the seminal work of Barles and Souganidis (1991) to stochastic differential games. Using our numerical scheme, we then compute derivative prices in a general strategic framework and present comparative statics results.

The present work contributes to the literature on derivative pricing with market frictions, and in particular to the branch of this literature focusing on non–Walrsian trading due to price impact of large investors. Early contributions such as Frey and Stremme (1997), Frey (1998), Platen and Schweizer (1998) and Bank and Baum (2004) focused on dynamic replication based hedging strategies and the resultant feedback effect on underlying security prices.

Later works such as Bouchard, Loeper and Zou (2016), Bouchard, Loeper and Zou (2017), Bank, Soner and Voß (2017), Loeper (2018) examine replication based pricing in an extended Black–Scholes framework with price impact, while Kraft and Kühn (2011), Guéant and Pu (2017), Dolinsky and Moshe (2022) and Ekren and Nadtochiy (2022) focus on utility based indifference pricing of derivatives with non–Walrasian trading. We generalize the framework considered in these works, which often rely on simplifying assumptions such as temporary price impact or uniformly bounded strategies, and focus on the singular hedging problem which arises on account of permanent price impact.

Hedging problems with permanent price impact and unbounded strategies have been studied in the context of utility based option pricing in Lions and Lasry (2007) using the diffeomorphic flow approach to singular control problems introduced in Lasry and Lions (2000), in Bouchard, Loeper, Soner and Zhou (2019) using a stochastic target approach to tackle the problem of (super) replication pricing of derivative securities, in Bouchard and Tan (2022) through a dual formulation focusing on hedging of path dependent options, as well as Fukasawa and Stadje (2018). These works focus exclusively on a monopolistic framework with a single large investor, serving as important precursors of our work. We generalize their setup by considering utility based derivative pricing in an oligopolistic framework with more than one large investor, focusing on strategic competition for liquidity absent from models with a single large investor. This has an important bearing on the indifference price of a derivative as well as the optimal hedging strategy of a large investor since it factors in the price impact of other large investors.

There is a growing body of literature dealing with pricing of derivatives in markets with multiple large investors, however, our work differs in important respects from existing works. Carvajal (2018) focuses on arbitrage pricing in markets organized as a batch auction market, with no noise traders, in a two–period setup. Nyström and Parviainen (2017) and Aïd, Callegaro and Campi (2020) consider derivative pricing and dynamic hedging in a differential game framework, with the former limiting attention to a zero sum game and the latter focusing on commodity option pricing where only a single agent can influence price drift. In contrast, we highlight limitations of arbitrage pricing in markets with strategic competition for liquidity such as a limit order book with noise traders and explore derivative pricing in a non–zero sum singular stochastic differential game framework where each large investor can potentially influence price drift.

More broadly, our work contributes to the literature on asset pricing with convex frictions with notable contributions including Prisman (1986), Ross (1987), Jouini and Kallal (1995), Luttmer (1996), Jouini and Kallal (1999) and Lécuyer and Martins-da Rocha (2021) which variously deal with issues related to viability, arbitrage and pricing in markets with convex frictions such as taxation in a general equilibrium framework. We instead focus on pricing derivative securities in a relative pricing framework with liquidity frictions arising out of price impact in markets with non–Walrasian trading.

As a coda to the introduction, we briefly sketch the layout of the present work. In Section 2 we analyze a stylized two–period, binomial motivating example with the aim of illustrating the limitations of arbitrage pricing in the case of non–Walrasian trading by large investors. In Section 3, we set up a continuous time model which extends the canonical Black–Scholes framework by introducing a Cournot duopoly in the underlying market and introduces the idea of indifference pricing when investors compete strategically for liquidity. Section 4 is devoted to the illustration of the singular nature of an investor's optimization problem and a heuristic construction of the auxiliary optimization problem. In Section 5, we formally state and prove the equivalence of an investor's singular optimization problem and the constructed auxiliary optimization problem. Section 6 builds on the equivalence result to analyze strategic indifference pricing of derivatives analytically as well as numerically, while Section 8 concludes.

2. Two–Period Motivating Example

This section illustrates the principal motivation underlying the present work in a stylized binomial or Cox–Ross–Rubinstein framework.[3] Specifically, we present a motivating example which illustrates breakdown of the replication pricing approach, based on law of one price under no arbitrage, in a two–period binomial framework with price impact. In subsequent discussion, we use the terms derivative security, option and contingent claim interchangeably, in the sense of a financial asset whose payoff depends on the future price of other underlying assets and/or the uncertain state of nature. Our approach is firmly rooted in the *relative pricing* tradition, ubiquitous in the option pricing literature (Cochrane, 2009, Chapter 17), where we take the price process of the underlying assets as given and focus our attention on determining the value of a derivative security given those prices.

To begin with, recall that in a *complete* market system the collection of underlying financial assets suffices to span all possible contingencies by definition. Consequently, one can obtain a *unique* price for a derivative security under the minimal assumption of absence of arbitrage opportunities.[4] In practice, the derivative security price is computed primarily through the replication approach, which involves imitating the derivative payoff via a portfolio of underlying assets, where the existence of a replicating portfolio is ensured by the completeness of underlying financial market. If there are no arbitrage opportunities, the price of the derivative security must necessarily equal the price of constructing the replication portfolio. Alternatively, one can use the no arbitrage principle to construct a unique risk–neutral or martingale probability measure (equivalently, a unique stochastic discount factor) such that the price of a contingent claim equals its expected discounted payoff with respect to the risk–neutral measure.

To fix ideas, we present below an example demonstrating derivative pricing in a standard two–period binomial model. Suppose that there are only two assets in the economy – a risk–free asset and a risky asset. Let the risk–free rate of return be zero and the price of the risk–free asset be normalized to one. The evolution of the price of the risky asset is depicted below

In Figure 1, price of the risky asset at date zero, denoted by S_0 is assumed to be deterministic and equals 10, while risky asset price at date one is assumed random and either moves up to $S_1^{\uparrow} = 20$ or drops down to $S_1^{\downarrow} = 5$, so that risky asset return equals either $u = 2$ or $d = 0.5$. We are required

[3]The name acknowledges the authors of the seminal work Cox, Ross and Rubinstein (1979), who introduced the framework as a discrete–time analogue of the seminal Black–Scholes option pricing model.

[4]Note that our usage of the term *absence of arbitrage* follows Back (2010, Section 4.1) and Campbell (2017, Section 4.2.2). Specifically, we consider the absence of arbitrage opportunities to be a stronger condition than than the law of one price, unlike Cochrane (2009, Section 4.1), and thus the law of one price is implicit in our assumption of no arbitrage opportunities.

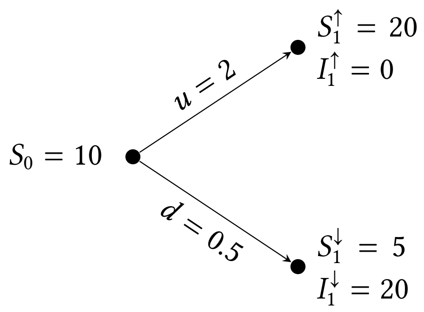

Figure 1: Derivative Pricing In Two–Period Binomial Model

to price a state contingent claim identified with its random date 1 payoff I_1, which can be either $I_1^{\uparrow} = 0$ or $I_1^{\downarrow} = 20$. Following standard arguments, we aim to construct a replicating portfolio at date 0 consisting of R_0^f units of the risk–free asset and Δ_0 units of the risky asset, by solving the following pair of equations

$$R_0^f + uS_0\Delta_0 = I_1^{\uparrow}$$
$$R_0^f + dS_0\Delta_0 = I_1^{\downarrow}$$

It is straightforward to check that the replication portfolio consists of risk–free holding $R_0^f = 80/3$ and risky asset position $\Delta_0 = -4/3$. Assuming a representative investor with zero initial endowment of underlying assets and absence of arbitrage opportunities, the law of one price ensures that price of the contingent claim at date zero, denoted by C_0 equals the price of constructing the replication portfolio, that is $I_0 = R_0^f + \Delta_0 S_0 = 40/3$.

From the example above, it is evident that in the standard two–period binomial model so long as $u \neq d$, the *payoff space* Ξ – which is defined to be the locus of vector of payoffs across the two states that can be replicated via a portfolio of underlying securities – equals the linear span of underlying security payoffs, that is, $\Xi = \mathbb{R}^2$. Importantly, the complete market paradigm abstracts from informational as well as institutional market frictions. In the presence of market frictions such as transaction costs, of which non–Walrasian trading on account of non–negligible price impact by large investors forms a particular case, financial markets are no longer complete in the sense of spanning every possible contingency as we illustrate below.

To this end, we modify the standard two–period binomial setup by relaxing the assumption of price–taking investors. Specifically, we consider a single large investor whose trading influences the price of the underlying risky asset. Thus, if S_0 denotes the unperturbed price or the mid–price of the risky asset at date zero, where following Glosten and Milgrom (1985) S_0 is interpreted as the consensus value of the risky asset at date 0 given all publicly available information,[5] and Δ_0 denotes the risky asset holding of the large investor, then traded price of the risky asset is assumed to be given by $\hat{S}_0(S_0, \Delta_0)$. The price of the risky asset at date one is assumed to be stochastic that can either take the value $u\hat{S}_0$ or $d\hat{S}_0$, corresponding to return of $u > 1$ or $d < 1$ respectively.

Subsequently, we work under the simplifying assumption that $\hat{S}_0 = S_0 + \lambda S_0 \Delta_0$, where S_0 is deterministic as above and λ is a positive constant, which serves as an analogue of *Kyle's Lambda* (Kyle, 1985), and represents the price impact parameter of the large investor. Intuitively, λ^{-1} measures

[5]Note that this formulation is characteristic of rational expectations models of financial markets with a single risky asset since the pioneering work of Grossman (1976).

the depth of the market relative to the holding of the large investor, that is, λ represents the relative change in the price of the risky asset caused by a unit holding on part of the large investor. Alternatively, given that we assume zero initial endowment, the term $\lambda S_0 \Delta_0$ captures the impact of large investor's order–flow on the traded price of the risky asset.

In the modified binomial setup with a single large investor described above, it may no longer be feasible to construct a portfolio of underlying assets which replicates the risk profile of a given contingent claim. In other words, it is no longer true that $\Xi = \mathbb{R}^2$. To see this, recall the claim I_1 whose payoff profile across the two states was given by $I_1^{\uparrow} = 0$ and $I_1^{\downarrow} = 20$. Further, suppose that the market depth parameter $\lambda = 0.2$, which implies that risky asset price evolution is as depicted in Figure 2 below

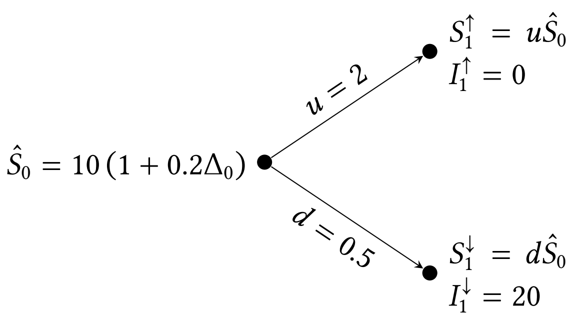

Figure 2: Two–Period Binomial Model With Price Impact

If we suppose that the claim is settled at maturity via physical delivery at market value, that is, there is no price impact at date 1, then in order for a portfolio consisting of R_0^f units of the risk–free asset and Δ_0 units of the risky asset to replicate the risky payoff profile of the contingent claim I_1, R_0^f and Δ_0 must solve the system of equations (2.1). Upon eliminating R_0^f from the pair of equations (2.1) we are lead to a (quadratic) equation in Δ_0 which has no real–valued solution, ruling out the existence of a replicating portfolio.

$$\begin{aligned} R_0^f + 20\Delta_0\,(1 + 0.2\Delta_0) &= I_1^{\uparrow} = 0 \\ R_0^f + 5\Delta_0\,(1 + 0.2\Delta_0) &= I_1^{\downarrow} = 20 \end{aligned} \tag{2.1}$$

Intuitively, in order to replicate the risk associated with the claim I_1, a replicating portfolio (if it exists) must necessarily provide the investor with an identical payoff profile. However, if an investor has non-negligible price impact, the expected payoff associated with a position Δ in the risky asset is no longer monotonic in Δ_0. In fact, given our assumption regarding the functional form of $\hat{S}_0(S_0, \Delta_0)$, the expected payoff from investing in the risky asset is strictly convex in Δ_0 with an interior minima thereby precluding arbitrarily low payoffs, even in the absence of portfolio constraints. This observation is formalized in Figure 3 below which depicts the payoff space Ξ for the binomial model with a single large investor considered above, defined as the affine half–space $\Xi = \left\{(I^{\downarrow}, I^{\uparrow}) \in \mathbb{R}^2 \,|\, 225 + 12\,(I^{\uparrow} - I^{\downarrow}) \geq 0\right\}$[6]

[6]It may seem that this limitation arises on account of the fact that price of the risky asset becomes unbounded. Interested readers are directed to the appendix where we consider an alternate example with bounded prices to show that this line of argument is incorrect.

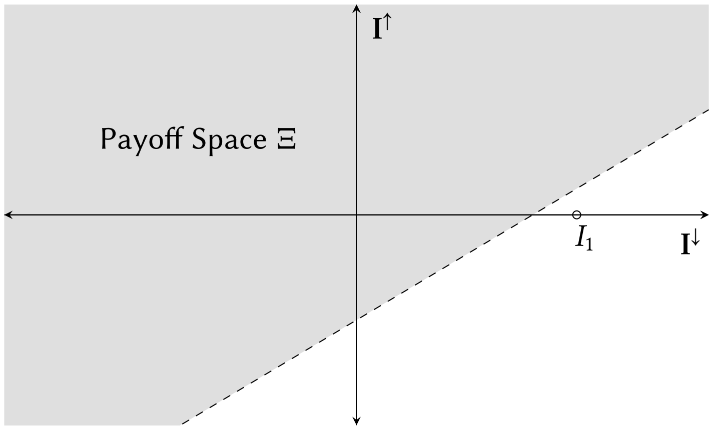

Figure 3: Payoff Space In Binomial Model With Price Impact

Note: The figure plots the payoff space Ξ, represented by the shaded area, for a two–period binomial model with price impact, with date 0 unperturbed price $S_0 = 10$, the return in the *good* state of the world $u = 2$, the return in the *bad* state of the world $d = 0.5$, market depth parameter $\lambda = 0.2$, and traded price $\hat{S}_0 = S_0\,(1 + \lambda\Delta_0)$. Note that the payoff space Ξ is a strict subset of $\mathbb{R}^2$, and that the claim I_1 with payoff structure $(0, 20)$ belongs to $\Xi^c = \mathbb{R}^2 \setminus \Xi$.

It is a well–known fact in asset pricing that so long as the law of one price holds, there exists at least one discount factor (which may not be strictly positive) that can be used to price uncertain streams of payoffs associated with any contingent claim. However, when financial markets are incomplete, one fails to obtain a *unique* stochastic discount factor in general (alternatively, a unique equivalent risk–neutral/martingale probability measure) even under the seemingly mild and natural assumption of no arbitrage, see Campbell (2017, Section 4.2). Consequently, unless the claim payoff belongs to Ξ, it cannot be priced *uniquely* by appealing to no arbitrage arguments or equivalently, by assuming that the stochastic discount factor is strictly positive (Föllmer and Schied, 2016, Corollary 1.35).

This problem is further intensified when market incompleteness is induced by market frictions since it typically leads to a nonlinear payoff space Ξ, as well as a nonlinear replication pricing functional on Ξ, as can be inferred from the example above. Due to these nonlinearities, the replication price of a contingent claim with payoff in Ξ will differ in general from its no arbitrage price derived through a stochastic discount factor, on account of deadweight loss generated by market frictions, unlike frictionless incomplete markets, see Jouini and Kallal (1999, Section 2) and Cochrane (2009, Section 4.2) for a more comprehensive discussion.

Despite significant interest and advances in theoretical and practical aspects of derivative pricing since the pioneering contributions of Black and Scholes (1973) and Merton (1973), there is little agreement regarding a suitable principle for pricing and hedging of derivative securities in incomplete markets, particularly when arguments based on absence of arbitrage opportunities are insufficient to pin down a unique price. In a lighter vein, while surveying recent developments in mathematical finance and financial economics in the context of option pricing, Jacod and Protter (2017) refer to the problem of contingent claim pricing and hedging in incomplete markets as *the largest elephant in the room.*

3. Dynamic Indifference Pricing and Strategic Hedging

In this section we analyze preference based indifference pricing of derivative securities in a continuous–time setup where an investor holding a nonzero derivative security position can hedge the risk associated with the derivative position by trading an underlying set of *primary* financial assets dynamically via market orders in an order–book like setup where the traded price of underlying assets is influenced by the order flow of multiple large investors, including the investor hedging the derivative position.

A typical example of the framework described above as discussed in Alexander, Chen and Imeraj (2023) is the crypto derivative assets market which has witnessed increased participation from large proprietary trading firms, major traditional financial conglomerates and top–tier investment banks. In order to hedge the risk associated with an open–interest crypto derivative listed on a crypto exchange such as Deribit, whose margin call and settlement price are quoted in terms of a cryptocurrency like Bitcoin, Ethereum or Solana, investors must trade the relevant underlying cryptocurrency using a traditional fiat currency. However, crypto–fiat currency trading pairs are highly illiquid on account of the fragmented nature of liquidity associated with cryptocurrencies, due to which the traded price dynamics are appreciably influenced by order flow of investors.

Throughout this section, we maintain the standing assumption of an underlying stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with respect to which all random variables and stochastic processes shall be defined. We assume that the stochastic basis supports a one–dimensional standard Brownian motion $B = \{B_t\}_{t\in[0,T]}$, and satisfies the usual conditions of $\mathbb{P}$–completeness as well as right–continuity of the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. For simplicity, we also assume that $\mathcal{F}_T = \mathcal{F}$. In the following subsection, we provide a formal description of the financial market framework.

3.1 Financial Market Model

We consider a stylized model of a financial market consisting of two primary assets, where one of the primary assets is a risk–free asset, whose rate of return at time t is denoted by r_t. Investors can also trade a single risky asset using market orders via an order–book operated by a decentralized exchange or a market maker within a centralized exchange. There are two institutional investors in the financial market, indexed by $i \in \mathcal{I} = \{-1, 1\}$, who invest in primary assets and share a common investment horizon, $T \in (0, \infty)$.

We assume that the institutional investors face no leverage constraint and the risk–free rate of return remains unaffected by the trading of these investors. We let Δ_t^i represent the number of units of the risky asset held by investor i and x_t^i represent the trading rate of investor i at time $t \in [0, T]$. We suppose that the initial level of stock holdings Δ_0^i is given (deterministic) for $i \in \mathcal{I}$, and that the portfolio dynamics of ith institutional investor are characterized by the following differential equation

$$d\Delta_t^i = -x_t^i \, dt \tag{3.1}$$

From the dynamics above that it follows that $x_t^i > 0$ implies that investor i holds an instantaneous selling position in the stock, while $x_t^i < 0$ implies an instantaneous buying position in the stock on part of investor i. Subsequently, we work under the simplifying assumption that the risk free rate r_t is identically zero for all t, or equivalently, we assume that the risk–free asset acts as a numéraire.

We turn our attention to the crucial issue of modelling the instantaneous impact of trading by ith investor on the price of the risky asset. As a clarifying remark, we note that we limit attention to permanent price impact and make no provision for any reversion towards the pre–trade price subsequent to the execution of an institutional investor's order flow. This implies that we do not consider slippage costs or other transaction costs associated with trading.

While our focus on the conceptually simple case of permanent price impact due to market orders facilitates tractability, it also carries operational significance since hedging algorithms conventionally employ a strategic layer for determining the optimal hedging schedule, accounting for permanent price impact costs, followed by routing of computed orders via a tactical layer which internalizes temporary price impact costs, see Guéant (2016, Section 3.1, Page 41). In order to incorporate the impact of trading by ith investor on the traded price of the risky asset, we assume that the dynamics of the stock price process S are governed by the following stochastic differential equation

$$dS_t = \sigma\, dB_t + \sum_i \Theta^i\left(x_t^i\right) dt; \text{ with } \sigma > 0 \tag{3.2}$$

The function $\Theta^i(\cdot) : \mathbb{R} \to \mathbb{R}$ in the equation above captures the effect of the order flow of investor i on the traded price of the risky asset, and the positive constant σ denotes the exogenous volatility of the risky asset price. It is immediate from above that we make two simplifying assumptions – first, we assume the order flow of institutional investors affects only the drift of the price process of the risky asset and second, the price impact of the two institutional investors admits a simple additive representation. We also assume that initial price of the risky asset S_0 is deterministic and that the exogenous drift of the risky asset price process is zero, which is typical of the literature on optimal execution given its focus on high–frequency trading where in the unperturbed risky asset price is modelled as a martingale.[7]

Since investors trade primary assets with a view to hedge the risk associated with their derivative position, we seek an economically well–founded criterion associated with a portfolio of primary assets held by an investor which is linked to the style or form of permissible derivative settlement.[8] To this end, we employ the cash flow generated on account of buying and selling of primary financial assets by an investor as the relevant measure of primary asset portfolio value, with W_t^i denoting the value of cash account employed in trading of primary assets by investor i at time t. Given the assumption of zero risk–free rate, the dynamics of cash account process W^i are governed by the following stochastic differential equation

$$dW_t^i = -S_t\, d\Delta_t^i = S_t\, x_t^i\, dt \tag{3.3}$$

We assume that W_0^i the cash account value at the initial time is given. In line with the definition above, we focus on the case of cash settled derivatives. It follows from the definition above that when the functions Θ^1 and Θ^2 are identically zero, the cash account value process corresponds to the canonical wealth process in the Black–Scholes framework.[9] A remark is in order concerning hedging portfolios satisfying (3.3) and the self–financing condition for hedging portfolios considered in the classical Black–Scholes hedging problem. While extending the notion of a self–financing hedging strategy in the context of a financial market with price impact is not straightforward, nevertheless,

[7]The results in this paper can be generalized in a straightforward manner to the case when the exogenous drift of the risky asset price is a nonzero scalar.

[8]Note that unlike the Black–Scholes (Bachelier) framework the liquidation value of the risky asset does not equal the product $\Delta_t^i\, S_t$ on account of the price impact terms.

[9]Another plausible justification for the definition above follows from the fact that in order to obtain the exact composition of ith investor's portfolio in terms of risk–free and risky asset holdings, it suffices to track the pair $\left(W^i, \Delta^i\right)$.

by requiring that hedging portfolios satisfy (3.3) we ensure that any change in the cash account value does not involve external flows of risk–free or risky financial asset, consistent with the fundamental principle underlying the self–financing condition.

While examining utility based prices for derivative securities, it is desirable to exclude arbitrage opportunities in the underlying financial market, as an essential prerequisite for the derivative prices to be compatible with the notion of economic equilibrium, since otherwise a rational investor would exploit the arbitrage opportunity to improve the value (utility) associated with the hedging portfolio without incurring additional risk leading to failure of optimality.[10]

However, conventional notions of an arbitrage trading strategy, such as a *free lunch with vanishing risk* defined in Delbaen and Schachermayer (1994), implicitly rely on the assumption of Walrasian trading by *small* investors who are price–takers and thus do not internalize the price impact of their trading while determining optimal portfolio holdings. Notably, this feature is persistent in notions which consolidate the idea of a classical arbitrage opportunity to derive improved bounds on option prices in incomplete markets such as a *good–deal* advocated by Cochrane and Saá-Requejo (2000) which augments no–arbitrage principle with additional Sharpe ratio restrictions, and no–arbitrage trading strategies with excessively favorable gain–loss ratio, termed *approximate arbitrage* by Bernardo and Ledoit (2000).

While seeking a cogent notion of viability of asset prices consistent with the principle of economic equilibrium attained as a consequence of optimal behavior by rational, non–Walrasian economic agents, one should ensure absence of profitable price manipulation by large investors who factor in their price impact while determining optimal trading strategy, in addition to classical arbitrage opportunities. This insight motivated the notion of *quasi–arbitrage* introduced by Huberman and Stanzl (2004), defined as *round–trip*[11] trading strategies generating infinite *expected* profits with an infinite Sharpe ratio. Later, Gatheral (2010) generalized the notion of quasi–arbitrage to *dynamic arbitrage*, defined as a round–trip trading strategy generating *positive* expected profits, thus defining price–manipulation in a broad sense. In the present work, we employ a strategic version of price manipulation criterion introduced in Gupta and Jacka (2023), whose definition is recalled below, as the relevant viability criterion. Note that given $i \in \mathcal{I}$, we consider $-i \in \mathcal{I}\backslash\{i\}$ denotes i's opponent.

Definition 3.1. Consider a $\mathbb{F}$–stopping time κ such that κ is bounded above by T, an adapted trading process $\{x_t^{-i}\}_{t\in[0,\kappa]}$ for investor $-i$, and suppose that the stopped process W_κ^i denotes the value of cash account of investor i at time κ. The adapted trading process $\left\{x_t^i\right\}_{t\in[0,\kappa]}$ defines a *profitable price manipulation* for investor i if the following conditions hold

(i) $\left\{x_t^i\right\}_{t\in[0,\kappa]}$ is a round–trip trade, that is, $\int_0^\kappa x_t^i\,dt \;=\; 0.$

(ii) $\mathbb{E}\left[W_\kappa^i\right] \;>\; W_0^i \;+\; \mathbb{E}\left[\int_0^\kappa \left(S_0 + \sigma B_t + \int_0^t \kappa^{-i}\left(x_u^{-i}\right)du\right) x_t^i\,dt\right]$

In view of the definition above, we remark that conditions defining a profitable price manipulation or dynamic arbitrage are strictly weaker than those describing a classical arbitrage, since profitable price manipulation opportunities only yield positive profit *on average* and are not required to be scalable, see Gatheral (2010) for further discussion. Nevertheless, as discussed in Gupta and Jacka

[10]For an extended discussion regarding viability of contingent claim prices obtained via relative pricing approach as models of economic equilibrium, the interested reader is advised to refer Harrison and Kreps (1979), Jouini and Kallal (1999) and Loewenstein and Willard (2000).

[11]Intuitively, a round–trip trading strategy is one which involves no net change in the composition of the portfolio.

(2023), under certain conditions an investor may in principle replicate a dynamic arbitrage *ad infinitum* to generate an almost sure net positive gain in her cash account value thereby approximating a classical arbitrage opportunity.

It is conventional wisdom in financial economics that viability of asset prices is intricately linked to the structure of price impact function. For instance, Black (1995) reasons that economic equilibrium in a financial market with rational agents necessitates that entry of an order causes a price move proportional to order size. Huberman and Stanzl (2004) formalize this intuition by proving that asset prices consistent with absence of quasi–arbitrage opportunities are supported only by linear permanent price impact functions, while Gatheral (2010) establishes that linearity of permanent price impact function is both necessary and sufficient to rule out profitable price manipulation opportunities for a large investor in the sense of dynamic arbitrage.[12] The lemma below extends this result to the case of dynamic arbitrage for strategic markets and its proof follows *mutatis mutandis* from the proof of (Gupta and Jacka, 2023, Lemma III.1), thus we state it here without proof.

Lemma 3.1. The absence of profitable price manipulation opportunities for investor i, in the sense of Definition 3.1 is equivalent to the permanent price impact function Θ^i being linear.

Thus, in subsequent discussion we assume that $\Theta^i(x^i_t) = -\theta^i x^i_t$ where θ^i is a positive constant representing the permanent price impact associated with a unit sell order. The sign restriction on θ^i is in line with literature examining agency problems arising on account of conflict of interest between large shareholders and minority shareholders and their effect on the value of the firm. Notable contributions such as Shleifer and Vishny (1986), Pagano and Röell (1998), and Bolton and Von Thadden (1998) argue that the presence of multiple large investors helps in effective control and monitoring, reducing managerial rents, curbing expropriation by an individual large investor and thus benefits other investors by improving expected returns.

A positive value of θ^i is also consistent with the extensive empirical evidence documenting improvement in the value of a financial asset associated with increased control by multiple large shareholders in a non–cooperative framework, see Faccio, Lang and Young (2001), Maury and Pajuste (2005), Attig, Guedhami and Mishra (2008) and the references therein. Similarly, empirical works quantifying price impact using low frequency trade data such as Plerou, Gopikrishnan, Gabaix and Stanley (2002), and Kyle and Obizhaeva (2016) determine that the price impact function is linear,[13] while a linear price impact function is also consistent with microstrucutral models based on monitoring effects such as DeMarzo and Urošević (2006), and models of local price impact with predictable meta–order flow, see for example Nadtochiy (2022). In view of these facts, we rewrite the dynamics for the price process of the risky asset as

$$dS_t = \sigma\, dB_t - \sum_i \theta^i x^i_t\, dt \tag{3.4}$$

[12] The assumption of a linear permanent price impact function also allows us to sidestep the issue of determining an optimal split of a meta–order into constituent *child orders* with a view to minimize price impact costs, see Gatheral, Schied and Slynko (2012).

[13] Empirical works analyzing price impact using high frequency trading data such as Gabaix, Gopikrishnan, Plerou and Stanley (2006), and Kyle and Obizhaeva (2016) find evidence in favor a power law price impact function. However, as Nadtochiy (2022) observes, a power law price impact for a meta–order is consistent with a local linear price impact function.

3.2 Indifference Pricing With Strategic Competition For Liquidity

We suppose that each investor aims to maximize utility associated with her composite portfolio value at terminal time T, where the composite portfolio value consists of cash accrued through trading of primary financial assets as well as payoff associated with option holdings maturing at terminal time T. We assume that preferences of investors over their terminal composite portfolio value are represented by exponential utility function u^i, defined over the entire real line so as to facilitate computation of bid as well as ask prices corresponding to a given derivative security payoff. Specifically, if γ^i denotes the absolute risk–aversion coefficient of investor i, and W^i denotes the value of her cash account, then her utility corresponding to an endowment of λ units of claim C_T is given by

$$u^i\big(W^i_T + \lambda C_T\big) \;=\; -\frac{1}{\gamma^i}\exp\Big\{-\gamma^i\big(W^i_T + \lambda C_T\big)\Big\}; \;\; \text{where} \;\; \gamma^i > 0,\; \lambda \in \mathbb{R} \tag{3.5}$$

We turn our attention to defining an admissible hedging strategy for an investor in the present context. It is well understood that in continuous time financial markets, the definition of an admissible hedging strategy plays a crucial role in establishing the well–posedness of an investor's optimization problem and consequently the fair price of a derivative security. We formulate a precise statement of the class of admissible hedging strategies, feasible for an investor in the sense of satisfying (3.3), in the definition below.[14]

Definition 3.2. Consider a $\mathbb{F}$–stopping time κ such that κ is bounded above by T. The class of admissible hedging strategies with respect to initial time κ is denoted by $\mathcal{A}_\kappa$ and is affixed to be the collection of trading strategies $X = \{x_t\}_{t\in[\kappa,T]}$ which satisfy

(i) $x_t = 0$, for all $0 \le t < \kappa$

(ii) $x : \Omega \times [\kappa, T] \to \mathbb{R}$ is adapted with respect to $\mathbb{F}$

(iii) $\mathbb{E}\left[\exp\left\{v\int_\kappa^T |x_t|\,dt \int_\kappa^T |\hat{x}_t|\,dt\right\}\right] < \infty$, for all $v \in \mathbb{R}$, $\hat{X} \in \mathcal{A}_\kappa$

We briefly discuss each of the three conditions specified in Definition 3.2. The first condition is a normalization requiring that an investor commence trading in the risky financial asset at the initial time, which is justified in view of our focus on the problem of hedging and pricing of derivative securities, while the second condition is standard in models with symmetrically informed investors which stipulates that an investor can condition her choice of hedging strategy at time t on all the public information available up to and including time t. The third admissibility condition is in line with extant literature on strategic trading, see Back and Baruch (2004) and Carlin, Lobo and Viswanathan (2007), where an investor's collection of admissible strategies depends on the strategies chosen by her rivals[15] thereby ensuring that an investor's optimization (best–response) problem is well–defined in the sense of having finite indirect utility,[16] which is essential for the pricing and hedging problem to be non–trivial.

[14] Interested readers are directed to Schachermayer (2001) for an illuminating discussion regarding the delicate issue of admissibility in the context of exponential utility maximization in continuous time financial markets.

[15] Thus, in view of this fact the strategic hedging game we consider here should be seen as *generalized game* as in Back, Cao and Willard (2000), rather than a conventional game where the set of feasible strategies for each investor is exogenously specified.

[16] It is straightforward to check that the third admissibility condition also suffices to rule out the existence of doubling strategies for a non–Walrasian trader in the spirit of Back (1992).

In subsequent discussion, we denote the state vector associated with the optimization problem of investor i at time t as Y_t^i with $\left(Y_t^i\right)^{\mathcal{T}} = \left[S_t, \Delta_t^i, \Delta_t^{-i}, W_t^i, W_t^{-i}\right]$. Additionally, we introduce vector–valued functions $\mathbf{a}^i$, $\mathbf{b}^i$, and $\mathbf{v}^i$ below

$$\mathbf{a}^i\left(Y_t^i\right)^{\mathcal{T}} = \left[-\theta^{-i}, 0, -1, 0, S_t\right]$$

$$\mathbf{b}^i\left(Y_t^i\right)^{\mathcal{T}} = \left[-\theta^{i}, -1, 0, S_t, 0\right]$$

$$\mathbf{v}^i\left(Y_t^i\right)^{\mathcal{T}} = \left[\sigma, 0, 0, 0, 0\right]$$

With the help of the definition of the vector–valued functions $\mathbf{a}^i$, $\mathbf{b}^i$ and $\mathbf{v}^i$ introduced above we can characterize the dynamics of the controlled state process Y^i corresponding to the optimization problem of investor i in terms of the following multivariate stochastic differential equation

$$dY_t^i = \mathbf{a}^i\left(Y_t^i\right) x_t^{-i} dt + \mathbf{b}^i\left(Y_t^i\right) x_t^i\, dt + \mathbf{v}^i\left(Y_t^i\right) dB_t; \quad \text{with } Y_0^i = \left[S_0, \Delta_0^i, \Delta_0^{-i}, W_0^i, W_0^{-i}\right]^{\mathcal{T}} \tag{3.6}$$

The following proposition proves that admissibility conditions outlined in Definition 3.2 guarantee the existence of a unique, non–explosive, strong solution to the multivariate stochastic differential equation (3.6). This involves two principal challenges – first, as noted above the collection of admissible strategies for each investor is endogenously determined as a function of the strategy tuple chosen by other investors and second, the coefficients in (3.6) do not satisfy a Lipschitz hypothesis in general, rendering standard existence results such as those contained in Protter (2004, Section V.3) inapplicable.

Proposition 3.1. The system of stochastic differential equations (3.6) has a unique strong solution, and the solution is non–explosive, that is, the lifetime of the solution, $\liminf_{n\to\infty} \left\{t > 0,\ \left|Y_t^i\right| \geq n\right\} > T$, $\mathbb{P}$-almost surely.

Proof. The claim follows from the proof of (Gupta and Jacka, 2023, Proposition III.1). □

Given a suitable definition of admissible hedging strategies, we define the utility indifference price of a derivative security in a mathematically rigorous fashion.[17] To this end, we recall that the utility indifference bid (buy) price $C_\kappa^{B,i}(\lambda)$ at time κ associated with $\lambda > 0$ units of contingent claim C_T is defined to be the value (measured in terms of the units of the risk–free asset) which makes an investor indifferent in terms of her expected utility under Nash equilibrium trading, between paying nothing and not having λ units of the claim C_T and paying $C_\kappa^{B,i}(\lambda)$ at the initial time κ while receiving the corresponding payoff λC_T at time T. In other words, the investor is willing to pay at most $C_\kappa^{B,i}(\lambda)$ units of the risk–free asset for λ units of the claim C_T at time T.[18]

Two remarks are in order concerning the definition above – first, the indifference bid price $C_\kappa^{B,i}(\lambda)$ for investor i depends in general not just on their own initial endowment of the risk–free and risky asset but also on the initial endowment of their rival investor, in addition to the derivative position of their rival. Thus, in subsequent discussion, we shall assume that investor $-i$ receives zero net endowment of the derivative claim C_T to facilitate tractability and avoid cluttered notation. Second,

[17] Utility indifference prices are also commonly referred to as reservation prices in the literature, see Munk (1999). Alternatively, works such as Detemple and Sundaresan (1999) use the term private valuation to emphasize the investor specific aspect of the computed price.

[18] Note that here we implicitly assume the existence of a unique Nash equilibrium wherever required for the ease of exposition. We defer a rigorous discussion of this issue till later to avoid a thicket of technicalities at this stage in favour of building intuition.

in order to compute the indifference price of a derivative claim under strategic competition for liquidity, we need to compute Nash equilibrium payoffs corresponding to two stochastic differential games.

The first corresponds to Merton–Cournot portfolio choice game, analyzed in Gupta and Jacka (2023), when investor i too has a zero position in the claim. For completeness, we recall here the definition of the best–response value function of investor i associated with the Merton–Cournot stochastic differential game. Given a deterministic initial state vector $\left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}$ and an admissible trading strategy $X^{-i} \in \mathcal{A}_\kappa$ for investor $-i$, the best–response value function of investor i, denoted as $J^{i,0}$ with the zero in the superscript indicating that investor i has zero position in the contingent claim, equals the supremum of expected utility from the terminal value of her cash account, where the supremum is taken over all admissible strategies $X^i \in \mathcal{A}_\kappa$. Specifically,

$$J^{i,0}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right) = \sup_{X^i \in \mathcal{A}_\kappa} \mathbb{E}\left[u^i\left(W^i_T\right)\right]$$

If the supremum in the expression above is attained by some admissible strategy $\hat{X}^i \in \mathcal{A}_\kappa$ of investor i, we define the associated indirect utility functional $U^{i,0}\left(\hat{X}^i; X^{-i}\right)$ for investor i as investor i's expected utility when investor i selects an admissible best–response to X^{-i}. Formally, we have

$$U^{i,0}\left(\hat{X}^i; X^{-i}\right) = J^{i,0}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)$$

Similarly, a strategy tuple $\left(\hat{X}^{-1}, \hat{X}^1\right)$ is a Nash equilibrium for the Merton–Cournot stochastic differential game if $\hat{X}^{-1}, \hat{X}^1 \in \mathcal{A}_\kappa$ and

$$J^{i,0}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; \hat{X}^{-i}\right) = U^{i,0}\left(\hat{X}^i;\, \hat{X}^{-i}\right);\ \text{where } i \neq -i \text{ and } i, -i \in \mathcal{I}$$

We let $\hat{J}^{i,0}\left(\kappa, [S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa]^{\mathcal{T}}\right)$ denote the Nash equilibrium payoff of investor i in the Merton–Cournot stochastic differential game.

The second stochastic differential game we need to solve for in order to determine the utility indifference bid price for a contingent claim is the strategic investment game where exactly one investor has bought the claim, which we term Black–Scholes–Cournot stochastic differential game.[19] In subsequent analysis of the Black–Scholes–Cournot game, we assume without loss of generality, that investor $i \in \mathcal{I}$ holds a positive position in the claim while investor $-i$ trades only in the underlying primary financial assets and has zero position in the claim.

Since, investor $-i$ has no position in the claim we denote the best–response value function of investor $-i$ as $J^{-i,0}$, corresponding to a given deterministic initial state vector $\left[S_\kappa, \Delta^{-i}_\kappa, \Delta^i_\kappa, W^{-i}_\kappa, W^i_\kappa\right]^{\mathcal{T}}$ and an admissible hedging strategy $X^i \in \mathcal{A}_\kappa$ for investor i, which much like the Merton–Cournot portfolio choice game equals the supremum of expected utility from the terminal value of cash account, where the supremum is taken over all admissible strategies $X^{-i} \in \mathcal{A}_\kappa$. Formally, we have

$$J^{-i,0}\left(\kappa, \left[S_\kappa, \Delta^{-i}_\kappa, \Delta^i_\kappa, W^{-i}_\kappa, W^i_\kappa\right]^{\mathcal{T}}; X^i\right) = \sup_{X^{-i} \in \mathcal{A}_\kappa} \mathbb{E}\left[u^{-i}\left(W^{-i}_T\right)\right] \tag{3.7}$$

[19] Strictly speaking, the strategic hedging game should be referred to as Bachelier–Cournot game, given that the unperturbed price dynamics is driven by *arithmetic* Brownian motion like Bachelier (1900), and not *geometric* Brownian motion as in Black and Scholes (1973). Nevertheless, using an elegant chaos expansion argument Schachermayer and Teichmann (2008) illustrate that the theoretical option prices derived from the two specifications coincide quite well, and so we retain this terminology in view of its widespread use.

As earlier, if the supremum in the expression above is attained by some admissible strategy $\hat{X}^{-i} \in \mathcal{A}_\kappa$ of investor $-i$, the indirect utility functional $U^{-i,0}\left(X^i; \hat{X}^{-i}\right)$ for investor $-i$ is defined as investor $-i$'s expected utility when investor $-i$ selects an admissible best–response to X^i. Specifically,

$$U^{-i,0}\left(X^i; \hat{X}^{-i}\right) = J^{-i,0}\left(\kappa, \left[S_\kappa, \Delta_\kappa^{-i}, \Delta_\kappa^{i}, W_\kappa^{-i}, W_\kappa^{i}\right]^{\mathcal{T}}; X^i\right)$$

In the Black–Scholes–Cournot game investor i holds a positive position $\lambda > 0$ in the claim, while hedging the derivative by trading in the underlying primary financial market. Thus, the best–response value function of investor i in Black–Scholes–Cournot game is denoted as $J^{i,\lambda}$, where $\lambda > 0$ in the superscript is indicative of the fact that investor i has a nonzero long derivative position. As before, given a deterministic initial state vector $\left[S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}\right]^{\mathcal{T}}$ as well as an admissible hedging strategy $X^{-i} \in \mathcal{A}_\kappa$ for investor $-i$ the best–response value function of investor i is defined as the supremum of expected utility from the terminal value of cash account along with the derivative payoff, where the supremum is taken over all admissible hedging strategies $X^i \in \mathcal{A}_\kappa$. In view of this, we can write

$$J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}\right]^{\mathcal{T}}; X^{-i}\right) = \sup_{X^i \in \mathcal{A}_\kappa} \mathbb{E}\left[u^i\left(W_T^i + \lambda C_T\right)\right] \tag{3.8}$$

Further, if the supremum in the expression above is attained by some admissible hedging strategy $\hat{X}^i \in \mathcal{A}_\kappa$ of investor i, we define the indirect utility functional $U^{i,\lambda}\left(\hat{X}^i; X^{-i}\right)$ for investor i as investor i's expected utility when investor i selects an admissible best–response to X^{-i}. Thus,

$$U^{i,\lambda}\left(\hat{X}^i; X^{-i}\right) = J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}\right]^{\mathcal{T}}; X^{-i}\right)$$

Therefore, a strategy tuple $\left(\hat{X}^{-1}, \hat{X}^1\right)$ is a Nash equilibrium for the Black–Scholes–Cournot stochastic differential game if $\hat{X}^i, \hat{X}^{-i} \in \mathcal{A}_\kappa$, where $i \neq -i$ and $i, -i \in \mathcal{I}$ and

$$\begin{cases} J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}\right]^{\mathcal{T}}; \hat{X}^{-i}\right) = U^{i,\lambda}\left(\hat{X}^i;\ \hat{X}^{-i}\right) \\ J^{-i,0}\left(\kappa, \left[S_\kappa, \Delta_\kappa^{-i}, \Delta_\kappa^{i}, W_\kappa^{-i}, W_\kappa^{i}\right]^{\mathcal{T}}; \hat{X}^{i}\right) = U^{-i,0}\left(\hat{X}^{-i};\ \hat{X}^{i}\right) \end{cases}$$

It is not immediate if there exists a well–defined solution to the best–response problem of investor i, except of course in the trivial case when $\lambda = 0$ and the Black–Scholes–Cournot game reduces to the Merton–Cournot game, see Gupta and Jacka (2023). The rest of the paper focuses primarily on the analysis of best–response problem of investor i associated with the Black–Scholes–Cournot game and exploring the existence of Nash equilibria. Assuming that $\hat{J}^{i,\lambda}\left(\kappa, [S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}]^{\mathcal{T}}\right)$ denotes the utility of investor i under the Nash equilibrium[20] of the Black–Scholes–Cournot stochastic differential game, the utility indifference bid price $C_0^{B,i}(\lambda)$ associated with $\lambda > 0$ units of contingent claim C_T is then defined implicitly via certainty equivalent principle as the solution to

$$\hat{J}^{i,\lambda}\left(\kappa, [S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i} - C_\kappa^{B,i}(\lambda), W_\kappa^{-i}]^{\mathcal{T}}\right) = \hat{J}^{i,0}\left(\kappa, [S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}]^{\mathcal{T}}\right) \tag{3.9}$$

As a clarifying remark, we note that in defining the utility indifference bid price for investor i, we have assumed that investor $-i$ has zero position in the claim. In general the indifference price and optimal hedging strategy of investor i will depend upon the claim position of investor $-i$. We

[20]The definition can be extended to the case of multiple Nash equilibria by defining $\hat{J}^{i,\lambda}\left(\kappa, [S_\kappa, \Delta_\kappa^{i}, \Delta_\kappa^{-i}, W_\kappa^{i}, W_\kappa^{-i}]^{\mathcal{T}}\right) = \sup\left\{U^{i,\lambda}\left(\hat{X}^i;\ \hat{X}^{-i}\right) \mid \left(\hat{X}^i;\ \hat{X}^{-i}\right) \in \text{Nash Equilibria}\right\}$, or as $\inf\left\{U^{i,\lambda}\left(\hat{X}^i;\ \hat{X}^{-i}\right) \mid \left(\hat{X}^i;\ \hat{X}^{-i}\right) \in \text{Nash Equilibria}\right\}$ in the spirit of super–replication and sub–replication payoffs respectively.

can extend the definition to the case when the claim position of investor $-i$ is nonzero, albeit at the expense of tedious notation, and so we limit attention to the case when investor $-i$ has zero position in the claim as is the case for example when the claim C_T is a bespoke over the counter (OTC) derivative security. The subsequent analysis can be extended in a straightforward manner to the case when claim position of investor $-i$ is nonzero.

In order to extend the above framework to define the utility indifference ask (or sell) price $C_\kappa^{A,i}(\lambda)$ associated with selling of λ units of the derivative claim C_T by investor $-i$, we simply redefine the best–response value function of investor i in the Black–Scholes–Cournot game. That is, given deterministic initial state vector $\left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathcal{T}}$ along with an admissible hedging strategy $X^{-i} \in \mathcal{A}_\kappa$ for investor $-i$, the best–response value function of investor i in Black–Scholes–Cournot game is denoted as $J^{i,-\lambda}$, where $-\lambda < 0$ in the superscript is indicative of the fact that investor i has a nonzero short derivative position, and is defined to be the supremum of expected utility from the terminal value of cash account along with the claim payoff, where the supremum is taken over all admissible hedging strategies $X^i \in \mathcal{A}_\kappa$. Formally,

$$J^{i,-\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathcal{T}}; X^{-i}\right) = \sup_{X^i \in \mathcal{A}_\kappa} \mathbb{E}\left[u^i\left(W_T^i - \lambda C_T\right)\right]$$

If we let $\hat{J}^{i,-\lambda}\left(\kappa, [S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}]^{\mathcal{T}}\right)$ denote the Nash equilibrium payoff of investor i in the Black–Scholes–Cournot stochastic differential game, then the utility indifference ask price $C_\kappa^{A,i}(\lambda)$ associated with $\lambda > 0$ units of contingent claim C_T is defined implicitly via certainty equivalent principle as the solution to

$$\hat{J}^{i,-\lambda}\left(\kappa, [S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i + C_\kappa^{A,i}(\lambda), W_\kappa^{-i}]^{\mathcal{T}}\right) = \hat{J}^{i,0}\left(\kappa, [S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}]^{\mathcal{T}}\right) \tag{3.10}$$

Given the above definitions, we note that utility indifference bid and ask prices associated with trading λ units of contingent claim C_T for investor i are related by the well–known identity $C_\kappa^{A,i}(-\lambda) + C_\kappa^{B,i}(\lambda) = 0$, for $\lambda > 0$, see Henderson and Hobson (2009). In view of this relation, we focus our attention on solving for utility indifference bid price associated with $\lambda \in (-\infty, \infty)$ units of contingent claim C_T for investor i in subsequent analysis. The focal point of subsequent analysis will be the best–response problem of investor i in the Black–Scholes–Cournot game, since all other best–response problems follow as special cases.

In the present work, we follow Merton (1973) in formulating an investor's best–response problem as a stochastic optimal control problem and analyzing it through a dynamic programming approach. The choice of this *primal* approach necessitates the assumption of a Markovian setting, which allows characterization of an investor's indirect utility as a *viscosity* solution to a partial differential equation, the Hamilton–Jacobi–Bellman equation.

The choice of a Markovian setting also seems appropriate given that we consider a setup with permanent price impact where the strategic interaction between large investors is rooted in the influence their trading strategy has on the state vector. We recall that Markovian trading strategies are a class of closed–loop, feedback strategies where the path of the state process $\left\{Y_s^i : 0 \le s \le t\right\}$ up to time t influences trading rate x_t^i only through its time t value Y_t^i.[21] Formally,

[21] There are other variants of closed–loop trading strategies considered in the literature. See for example, Micheli, Muhle-Karbe and Neuman (2023) where an investor's trading rate is defined as a function of trading rates of other investors in a framework which focuses exclusively on temporary price impact.

Definition 3.3. Consider a $\mathbb{F}$–stopping time κ such that κ is bounded above by T. The class of (time–inhomogenous) *Markovian* hedging strategies with respect to initial time κ is denoted by $\mathcal{A}_\kappa^{\mathrm{m}}$ and it is defined to be the collection of trading strategies $X^i = \left\{x_t^i\right\}_{t\in[\kappa,T]}$ which satisfy

(i) X^i is admissible, that is, $X^i \in \mathcal{A}_\kappa$, for all $i \in \mathcal{I}$.

(ii) $x_t^i = x^i\left(t, Y_t^i\right)$, for all $\kappa \le t \le T$.

In view of our emphasis on dynamic programming approach, it is natural that we focus attention on Nash equilibrium in Markovian strategies, alternatively *Markov–Nash equilibrium* of stochastic differential games defined above. We allow for time–inhomogenous Markovian strategies given that the common investment horizon T is assumed to be finite. Though, it is well–known that in a finite player, finite horizon stochastic dynamic game with observable actions, a Nash equilibrium in Markovian strategies always exists, see Maschler, Zamir and Solan (2020, Theorem 15.16), the existence result is proved in a discrete–time framework where the action set of each player is finite or at least compact valued.

Nevertheless, the result provides a logical starting point of enquiry in our setup, particularly since in a continuous–time framework non–Markovian strategies may not be associated with a well–defined outcome path, see Fudenberg and Tirole (1991, Section 13.3.4) and the references therein for a discussion of technical issues associated with the existence of Nash equilibrium in stochastic differential games.

In subsequent discussion, we thus consider the best–response problem of an investor in a setup where trading strategies of both investors belong to the class of admissible Markovian strategies. Formally, given a deterministic initial state vector $\left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathcal{T}}$ the best–response value function $J^{i,\lambda}$ for investor i corresponding to an admissible Markovian strategy $X^{-i} \in \mathcal{A}_\kappa^{\mathrm{m}}$ of investor $-i$, is defined to be the supremum of expected utility over the set of all admissible Markovian strategies $X^i \in \mathcal{A}_\kappa^{\mathrm{m}}$,

$$J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathcal{T}}; X^{-i}\right) = \sup_{X^i \in \mathcal{A}_\kappa^{\mathrm{m}}} \mathbb{E}\left[u^i\left(W_T^i + \lambda C_T\right)\right]$$

4. Black–Scholes–Cournot Hedging Problem

In this section, we analyze the best–response problem of investor i in the Black–Scholes–Cournot stochastic differential game when $\lambda \neq 0$. To begin with, we illustrate singular nature of the best–response problem in the subsection below, which renders the partial differential equation approach to stochastic optimal control problems based on dynamic programming infeasible. The remainder of the section is devoted to a discussion of the diffeomorphic flow approach introduced in Lasry and Lions (2000) to deal with a particular class of singular stochastic optimal control problems, which serves as a remedy for the singularity of the best–response problem. Specifically, we extend the Lasry–Lions approach to utility based derivative pricing and hedging when there is strategic competition for liquidity among large investors.

Before proceeding to a detailed examination of these issues, a remark is in order regarding notation we employ in subsequent discussion. We denote first partial derivative of a function with respect to its qth argument by D_q and we denote the nth partial derivative of a function with respect to its qth argument for $n > 1$ as D_q^n. Further, given an initial time κ, we denote the initial state vector $[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa]^\mathcal{T}$ as Y^i_κ for the sake of brevity. Also, we let the operator $\mathcal{L}^x$ denote the infinitesimal generator associated with the diffusion (3.6) corresponding to constant control x, where recall that the infinitesimal generator is defined as

$$\mathcal{L}^x \phi\big(\kappa, Y^i_\kappa; X^{-i}\big) = \big\langle D_2\, \phi\big(\kappa, Y^i_\kappa; X^{-i}\big), \mathbf{a}^i\big(Y^i\big)\, x^{-i} + \mathbf{b}^i\big(Y^i\big)\, x \big\rangle + \frac{1}{2}\,\mathrm{tr}\left(D^2_2 \phi\big(\kappa, Y^i_\kappa; X^{-i}\big)\, \mathbf{v}^i\big(Y^i\big)\, \mathbf{v}^i\big(Y^i\big)^\mathcal{T}\right)$$

4.1 Singularity of Best–Response Hedging Problem

In order to illustrate the singular nature of the best–response problem of investor i in the Black–Scholes–Cournot game when $\lambda \neq 0$, we follow (Pham, 2009, Section 3.4) and let $\mathcal{H}$ denote the Hamiltonian associated with the best–response problem of investor i, which is defined below

$$\mathcal{H}\big(\kappa, Y^i_\kappa, D_2\, J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big), D^2_2 J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big)\big) = \sup_{x \in \mathbb{R}} \mathcal{L}^x J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big) \tag{4.1}$$

Note that the state space for the hedging strategy is $\mathbb{R}$, which in view of the equation above implies that the co–domain of the Hamiltonian $\mathcal{H}$ is the extended real line in general, as $\mathcal{H}$ may take the value $+\infty$ on its domain. It may seem tempting to address this issue by considering a bounded state space for the hedging strategy, however this is quite restrictive as it rules out many strategies of practical interest.

For example, suppose investor i wishes to unwind an initial position of $\Delta^i_0 > 0$ at time 0 to $\Delta^i_T = 0$ at the terminal time T using a simple time–weighted strategy $\Delta^i_t = \Delta^i_0 \sqrt{T-t}$ for $0 \leq t \leq T$. It is straightforward to check that $x^i_t = -d\Delta^i_t/dt$ fails to be bounded even in this simple case. Thus, we maintain the assumption that state space for the hedging strategy is $\mathbb{R}$ which results in singularity of the best–response problem due to which the Hamilton–Jacobi–Bellman equation associated with $J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big)$ is not well–defined in general. Nevertheless, observe that

$$\sup_{x \in \mathbb{R}} \mathcal{L}^x J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big) < \infty \;\text{ only if }\; \big\langle D_2\, J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big), \mathbf{b}^i\big(Y^i\big)\big\rangle = 0 \tag{4.2}$$

Thus, in order to characterize the indirect utility functional $J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big)$ associated with the best response problem of investor i in the Black–Scholes–Cournot game through the classical Hamilton–Jacobi–Bellman equation, one should ensure that the sufficiency condition above holds. This sufficiency condition lies at the root of diffeomorphic flow approach introduced by Lasry and Lions (2000) to analyze a particular class of singular stochastic optimal control problems. To see this, consider the integral flow $f^i\big(\varphi, y^i\big)$, parameterized by the scalar $\varphi \in \mathbb{R}$, derived from the controlled drift coefficient $\mathbf{b}^i$ as follows

$$\frac{\partial f^i(\varphi)}{\partial \varphi} = \mathbf{b}^i\big(f^i(\varphi)\big)\,,\;\text{ with } f^i\big(0, y^i\big) = y^i \tag{4.3}$$

Suppose that the best–response problem of investor i belongs to the class of singular stochastic optimal control problems whose indirect utility functional satisfies an invariance property with respect to the integral flow *à la* Lasry–Lions, that is, suppose we have $J^{i,\lambda}\big(\kappa, f^i\big(\varphi, Y^i_\kappa\big); X^{-i}\big) = J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big)$. We can differentiate both sides of the invariance equation with respect to φ and use the definition of f^i above (4.3) to see that it leads us to the sufficiency condition (4.2), which ensures a non–trivial characterization of $J^{i,\lambda}\big(\kappa, Y^i_\kappa; X^{-i}\big)$ in terms of the classical Hamilton–Jacobi–Bellman equation.

There are two principal advantages of Lasry–Lions approach. First, if a singular control problem satisfies flow invariance, one can construct an auxiliary stochastic control problem whose state space coincides with the quotient space induced by orbits of the diffemomorphic flow, such that the value function of the auxiliary problem is equivalent to that of the singular best–response problem. Second, the auxiliary control problem is tractable via standard dynamic programming methods by construction, which facilitates characterization of indirect utility of investor i corresponding to the best–response problem. In the following proposition, we establish the existence as well as certain regularity properties of the integral flow $f^i\left(\varphi, y^i\right)$ which plays a pivotal role in subsequent analysis, where the proof of the claim follows along the lines of Gupta and Jacka (2023, Proposition IV.1) and hence is omitted here.

Proposition 4.1. Given the multivariate differential equation (4.3), which determines the integral flow $f^i\left(\varphi, y^i\right)$ associated with the best–response problem of investor i in the Black–Scholes–Cournot stochastic differential game, we have

(i) There exists a unique solution $f^i\left(\varphi, y^i\right)$ to the integral flow equation above.

(ii) The function $f^i(\varphi, \cdot)$ is twice continuously differentiable for all $\varphi \in \mathbb{R}$.

4.2 An Auxiliary Hedging Problem

In this section we sketch the principal steps involved in the construction of auxiliary hedging problem, following the arguments in Lasry and Lions (2000). One can think of the auxiliary hedging problem as a *reduced* version of best–response problem of an investor, which transports the state variable instantaneously along the integral flow at no cost, through which we can solve for $J^{i,\lambda}$ by standard dynamic programming methods. Central to our construction is the integral flow $f^i(\cdot)$, derived from the controlled drift coefficient $\mathbf{b}^i$ associated with state dynamics of an investor's best–response problem, which can be computed in a straightforward manner by solving (4.3) to obtain

$$f^i\left(\varphi, Y^i\right) = \left[\mathbf{S} - \theta^i\varphi,\ \mathbf{\Delta}^i - \varphi,\ \mathbf{\Delta}^{-i},\ \mathbf{W}^i + \mathbf{S}\,\varphi - \frac{\theta^i}{2}\varphi^2,\ \mathbf{W}^{-i}\right]^{\mathcal{T}} \tag{4.4}$$

Note that the integral flow is linear in its initial condition, which is the state variable of the best–response problem of investor i. In this instance, a careful selection of the flow parameter φ in association with Itô's Lemma ensures the elimination of the drift terms responsible for singularity of the best–response problem of an investor from the dynamics of the state process of the best–response problem, thereby leading us to the dynamics of the auxiliary state process. Formally, given an initial time $\kappa \in [0, T]$ we consider the abridged process $\hat{Y}^i$ obtained by substituting the zero control, $x_t^i = 0, \forall\, t \in (\kappa, T]$ in the dynamics of the state process of the best–response problem of investor i (3.6), and replacing the initial condition with $f^i\left(-\varphi, Y_\kappa^i\right)$. That is,

$$\begin{aligned} d\hat{Y}_t^i &= \mathbf{a}^i\left(\hat{Y}_t^i\right) x_t^{-i} dt + \mathbf{v}^i\left(\hat{Y}_t^i\right) dB_t \\ \hat{Y}_\kappa^i &= f^i\left(-\varphi, Y_\kappa^i\right) \end{aligned} \tag{4.5}$$

Having thus defined the abridged process $\hat{Y}^i$, we next determine the flow f^i corresponding to the abridged process by replacing the state vector Y^i with the vector $\hat{Y}^i$ as the initial condition in

(4.4). To this end, suppose that the vector $\hat{Y}^i$ can be represented as $\hat{Y}^i = \left[\hat{S}, \hat{\Delta}^i, \hat{\Delta}^{-i}, \hat{W}^i, \hat{W}^{-i}\right]^{\mathcal{T}}$. Then, in view of (4.4) we can write the flow $f^i\left(\varphi, \hat{Y}^i\right)$ corresponding to the abridged process $\hat{Y}^i$ as

$$f^i\left(\varphi, \hat{Y}^i\right) = \left[\hat{\mathbf{S}} - \theta^i\varphi,\ \hat{\boldsymbol{\Delta}}^i - \varphi,\ \hat{\boldsymbol{\Delta}}^{-i},\ \hat{\mathbf{W}}^i + \hat{\mathbf{S}}\,\varphi - \frac{\theta^i}{2}\varphi^2,\ \hat{\mathbf{W}}^{-i}\right]^{\mathcal{T}} \tag{4.6}$$

We remark here that the abridged process plays a vital role in the construction of the auxiliary hedging problem as the state process for the auxiliary hedging problem is characterized via the abridged process through the integral flow $f^i\left(\varphi, \hat{Y}^i\right)$. Formally, in order to derive the auxiliary state process Z^i associated with the best–response problem of investor i in the Black–Scholes–Cournot stochastic differential game, we apply Itô's Lemma to (4.6), where the applicability of Itô's Lemma is immediate in view of Proposition 4.1. Further, given the linearity of flow f^i with respect to $\hat{Y}^i$, it is straightforward to check through routine computation that $D_2^2 f^i\left(\varphi, \hat{Y}_t^i\right) = \mathbf{0}_{5\times 5}$ which considerably simplifies the Itô's formula to yield the following for $t \in (\kappa, T]$

$$df^i\left(\varphi, \hat{Y}_t^i\right) = D_2 f^i\left(\varphi, \hat{Y}_t^i\right) \mathbf{a}^i\left(\hat{Y}_t^i\right) x_t^{-i} dt + D_2 f^i\left(\varphi, \hat{Y}_t^i\right) \mathbf{v}^i\left(\hat{Y}_t^i\right) dB_t \tag{4.7}$$

In the next step, we describe the dynamics associated with the controlled auxiliary state process with the help of the equation above. To this end, we define $Z^i = f^i\left(\varphi, \hat{Y}^i\right)$ and note by the time–shift property[22] of the flow that $\hat{Y}^i = f^i\left(-\varphi, Z^i\right)$ which we use to eliminate $\hat{Y}^i$ in the equation above. Further, by substituting the time variable φ with the adapted control process $\left\{\vartheta_t^i\right\}_{t\in[\kappa,T]}$ we arrive at the auxiliary state process as the solution of the resulting stochastic differential equation. In order to complete our description of the auxiliary control problem, it only remains to define the collection of admissible auxiliary hedging strategies. To this end, suppose that we are given a compact subset $\mathcal{D}$ of the real line $\mathbb{R}$, such that $0 \in \mathcal{D}$. We can then define the set of admissible hedging strategies for the auxiliary hedging problem as follows

Definition 4.1. Let κ be a $\mathbb{F}$–stopping time such that κ is bounded above by T. The class of admissible auxiliary hedging strategies with respect to initial time κ is denoted by $\mathcal{A}_\kappa^{\mathrm{a}}$ and it is defined to be the collection of hedging strategies $\mathcal{V} = \{\vartheta_t\}_{t\in[\kappa,T]}$ which satisfy

(i) $\vartheta_s = 0$, for all $0 \leq s < \kappa$

(ii) $\vartheta : \Omega \times [\kappa, T] \to \mathcal{D}$ is adapted with respect to $\mathbb{F}$

(iii) $\vartheta(\omega, \cdot)$ is left–continuous at T, for $\mathbb{P}$-almost every ω

Thus, having defined the class of admissible controls, the abridged process $\hat{Y}^i$ corresponding to constant zero control for the original controlled process, as well as the flow $f^i\left(\varphi, \hat{Y}^i\right)$ we are well equipped to define the controlled auxiliary state process along with the state dynamics associated with the auxiliary hedging problem below for $t \in (\kappa, T]$

$$dZ_t^i = D_2 f^i\left(\vartheta_t^i, f^i\left(-\vartheta_t^i, Z_t^i\right)\right) \mathbf{a}^i \circ f^i\left(-\vartheta_t^i, Z_t^i\right) x_t^{-i} dt + D_2 f^i\left(\vartheta_t^i, f^i\left(-\vartheta_t^i, Z_t^i\right)\right) \mathbf{v}^i \circ f^i\left(-\vartheta_t^i, Z_t^i\right) dB_t$$

Having sketched this broad general constructive argument we recall that in the specific context we consider in this paper the flow is linear with respect to its initial condition, which facilitates an

[22] See (Strogatz, 2018, Chapter 2) for a comprehensive treatment of the properties of the flow associated with a differential equation.

alternate equivalent derivation of the auxiliary state process directly through the flow associated with the original state process $f^i(\varphi, Y^i)$. This direct argument follows Goldys and Wu (2019) where it is shown that in the case of a linear flow, the construction of the auxiliary state process can be achieved by replacing the time variable in the flow $f^i(\varphi, Y^i)$ with $\mathbb{F}$–adapted process $\{\vartheta^i_t\}_{t\in[\kappa,T]}$ and using Ito's Lemma to determine the dynamics of auxiliary state process, so long as the dynamics of the process $\{\vartheta^i_t\}_{t\in[\kappa,T]}$ exactly cancel the drift terms responsible for singularity of the best–response problem. To see this, consider the terms of the flow (4.6) with the parameter φ replaced by the variable ϑ^i below

$$\pi^i = S - \theta^i\vartheta^i,\ \mu^i = \Delta^i - \vartheta^i,\ \mu^{-i} = \Delta^{-i},\ \psi^i = W^i + S\vartheta^i - \frac{\theta^i}{2}\left(\vartheta^i\right)^2,\ \psi^{-i} = W^{-i} \tag{4.8}$$

In order to define a controlled state process for the auxiliary hedging problem it remains to eliminate the undesirable drift terms which can be done by considering a *reduced* problem where we take the old state variable Δ^i as the new control process *à la* Lions and Lasry (2007) and a routine application of Itô's Lemma confirms the elimination of drift terms subsequent to this transformation. As a cautionary remark, we observe that the auxiliary state variables defined above are a mere notational convenience and should not be attributed any meaningful economic interpretation as such. In view of the equation above, we may then rewrite the controlled state process for the auxiliary hedging problem as $Z^i = \left[\pi^i, \mu^i, \mu^{-i}, \psi^i, \psi^{-i}\right]^{\mathcal{T}}$. Thus, it follows in view of (4.8) and (4.7) that the dynamics of the state process for the auxiliary control problem with $\Delta^i = \vartheta^i$ are then governed by the following system of stochastic differential equations with $t \in (\kappa, T]$

$$d\left[\pi^i_t, \mu^i_t, \mu^{-i}_t, \psi^i_t, \psi^{-i}_t\right]^{\mathcal{T}} = \left[-\theta^{-i},\ 0,\ -1,\ -\theta^{-i}\vartheta^i_t,\ \pi^i_t + \theta^i\vartheta^i_t\right]^{\mathcal{T}} x^{-i}_t\, dt + \sigma\left[1,\ 0,\ 0,\ \vartheta^i_t,\ 0\right]^{\mathcal{T}} dB_t$$

Further, in order to derive the initial condition we exploit the time–shift property of the flow in conjunction with the fact that $Z^i_\kappa = f^i(\varphi, \hat{Y}^i_s)$ and $\hat{Y}^i_\kappa = f^i(-\varphi, Z^i_\kappa)$ to obtain $Z^i_\kappa = Y^i_\kappa$. In addition, we also define vector–valued deterministic functions $\boldsymbol{\beta}^i$ and $\boldsymbol{\chi}^i$ as follows

$$\begin{aligned}
\boldsymbol{\beta}^i\left(\vartheta^i_t, Z^i_t\right) &= D_2 f^i\left(\vartheta^i_t, \hat{Y}^i_t\right)\mathbf{a}^i\left(\hat{Y}^i_t\right) = \left[-\theta^{-i},\ 0,\ -1,\ -\theta^{-i}\vartheta^i_t,\ \pi^i_t + \theta^i\vartheta^i_t\right]^{\mathcal{T}} \\
\boldsymbol{\chi}^i\left(\vartheta^i_t, Z^i_t\right) &= D_2 f^i\left(\vartheta^i_t, \hat{Y}^i_t\right)\mathbf{v}^i\left(\hat{Y}^i_t\right) = \sigma\left[1,\ 0,\ 0,\ \vartheta^i_t,\ 0\right]^{\mathcal{T}}
\end{aligned} \tag{4.9}$$

Next, we use the definition of the vector–valued functions $\boldsymbol{\beta}^i$ and $\boldsymbol{\chi}^i$ to rewrite the dynamics of the auxiliary state process Z^i succinctly as follows

$$\begin{aligned}
dZ^i_t &= \boldsymbol{\beta}^i\left(\vartheta^i_t, Z^i_t\right) x^{-i}_t dt + \boldsymbol{\chi}^i\left(\vartheta^i_t, Z^i_t\right) dB_t \\
Z^i_\kappa &= Y^i_\kappa
\end{aligned} \tag{4.10}$$

With the definition of controlled state process for the auxiliary hedging problem at our disposal, it remains for us to show that the system of stochastic differential equations governing its dynamics (4.10) admit a unique, non–explosive, strong solution given the admissibility conditions for auxiliary hedging strategies specified in Definition 4.1. The principal challenge involved in establishing this result is that the coefficients in (4.10) do not satisfy a Lipschitz condition, which implies that standard existence results such as those contained in Protter (2004, Section V.3) are inapplicable in the present context. We instead extend the Euler approximation method and localization argument employed in the proof of (Gupta and Jacka, 2023, Proposition V.1) to prove the existence of a unique, non–explosive, strong solution to the multivariate stochastic differential equation (4.10) in the proposition below.

Proposition 4.2. The system of stochastic differential equations (3.6) has a unique strong solution, and the solution is non–explosive, that is, the lifetime of the solution, $\liminf_{n\to\infty}\left\{t>0,\ \left|\mathrm{Z}_t^i\right|\geq n\right\}>T$, $\mathbb{P}$-almost surely.

Proof. The claim follows from the proof of (Gupta and Jacka, 2023, Proposition V.1). □

Lastly, in order to complete the construction of the auxiliary hedging problem, we need to specify the final remaining element – the indirect utility functional for the auxiliary hedging problem corresponding to the best–response problem of investor i in the Black–Scholes–Cournot stochastic differential game. To this end, we note that in the auxiliary hedging problem investor i aims to maximize utility associated with *transformed* composite portfolio value at terminal time T, where her choice of hedging strategies is limited to $\mathcal{A}_\kappa^{\mathrm{a}}$. Formally, given a deterministic initial auxiliary state $\left[S_\kappa,\Delta_\kappa^i,\Delta_\kappa^{-i},W_\kappa^i,W_\kappa^{-i}\right]^{\mathcal{T}}$, and an admissible strategy $X^{-i}\in\mathcal{A}_\kappa$ for investor $-i$, we define the indirect utility functional corresponding to the auxiliary hedging problem of investor i, denoted by $F^{i,\lambda}$ as

$$\begin{cases} F^{i,\lambda}\Big(\kappa,\left[S_\kappa,\Delta_\kappa^i,\Delta_\kappa^{-i},W_\kappa^i,W_\kappa^{-i}\right]^{\mathcal{T}};X^{-i}\Big) = \sup_{\mathcal{V}^i\in\mathcal{A}_\kappa^a}\mathbb{E}\left[\sup_{\boldsymbol{\varphi}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{\varphi},\psi_T^i\big)+\lambda C\circ f^i\big(-\boldsymbol{\varphi},\pi_T^i\big)\big)\right] \\ F^{i,\lambda}\Big(T,\left[S_T,\Delta_T^i,\Delta_T^{-i},W_T^i,W_T^{-i}\right]^{\mathcal{T}};X^{-i}\Big) = \sup_{\boldsymbol{\varphi}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{\varphi},\psi_T^i\big)+\lambda C\circ f^i\big(-\boldsymbol{\varphi},\pi_T^i\big)\big) \end{cases}$$

5. Invariance and Equivalence

We begin this section by establishing a verification result which proves that the best–response hedging problem of investor i indeed satisfies an invariance property with respect to the flow derived from the singular drift coefficient in (4.3). Subsequently, we show that the auxiliary hedging problem satisfies a similar invariance property. We then call upon these invariance properties in proving an equivalence result for the two stochastic optimal control problems which represents the focal point of this section. In subsequent analysis, we maintain the following standing assumption concerning the option payoff function C which is defined to be a function of S_T, the price of the risky financial asset at the terminal time.

Assumption 5.1. The function C, which denotes the payoff of a derivative security written on the risky financial asset with maturity date T, satisfies the following

(i) The function C is Lipschitz continuous in its argument, that is, there exists $K>0$ such that we have $\left|C(x)-C(y)\right|\leq K\left|x-y\right|$

(ii) Given admissible hedging strategy tuple $\left(X^i,X^{-i}\right)$, the random variable $C_T=C(S_T)$ has finite moment generating function at zero, that is, we have $\mathbb{E}\left[\exp\left(v\,C_T\right)\right]<\infty$, for all $v\in\mathfrak{R}$

We remark that the class of derivative securities satisfying (i) above is non–trivial as can be seen from the fact that it includes the ubiquitous European style put and call options, while also encompassing exotic derivatives such as the *chooser* options. We also underscore that the assumption is permissive with regard to securities such as state contingent claims to risk–free numéraire asset, whose payoff depends on the state of nature ω. The preceding assumption however is of limited value while analyzing utility based prices in the *manipulable* context, where investors are allowed to influence the option payoff through their price impact on the underlying asset at maturity.[23]

[23] Kumar and Seppi (1992) colloquially term such investor behaviour as *punching the close*.

Therefore, in order to extend our analysis to the manipulable case, we require the following additional assumption

Assumption 5.2. Given the flow f^i derived from singular drift coefficient $\mathbf{b}^i$ via (4.3), the payoff function C associated with a derivative security written on the risky financial asset with maturity T, satisfies the following for all $\left(\psi^i_T, \pi^i_T\right) \in \mathfrak{R}^2$

(i) The function $\varphi \longmapsto f^i\left(-\varphi, \psi^i_T\right) + \lambda C \circ f^i\left(-\varphi, \pi^i_T\right)$ attains its supremum at $\varphi^* \in (-\infty, \infty)$

(ii) The function $\varphi \longmapsto C \circ f^i\left(-\varphi, \pi^i_T\right) = C\left(\pi^i_T + \theta^i \varphi\right)$ is differentiable at φ^*

(iii) The function C is piecewise linear.

5.1 Best–Response Hedging Problem Invariance

We begin with an ancillary lemma which gathers few regularity properties of the indirect utility functional $J^{i,\lambda}$ associated with investor i's best–response hedging problem that serve as useful aids in proving the desired invariance result. Specifically, we establish that the indirect utility functional $J^{i,\lambda}$ is non–degenerate and possesses desirable continuity properties. The fundamental challenge in proving the lemma stems from our assumption concerning the preferences of investors, as the exponential utility function fails to be bounded when defined over the entire real line. Moreover, as stated earlier the best–response hedging problem of investor i is a singular stochastic optimal control problem, in view of which we first introduce a classical variant of the best–response hedging problem of investor i to tackle this issue. Formally, given a deterministic initial time $\kappa \in [0, T]$, and an admissible trading strategy $X^{-i} \in \mathcal{A}^{\mathrm{m}}_\kappa$ for investor $-i$, we define

$$J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\right) = \sup_{X^i \in \mathcal{A}^{\mathrm{m},n}_\kappa} \mathbb{E}\left[u^i\left(W^i_T + \lambda C_T\right)\right]$$

$$\text{with } \mathcal{A}^{\mathrm{m},n}_\kappa = \left\{X \in \mathcal{A}^{\mathrm{m}}_\kappa \,:\, |x_t| \leq n, \text{ for all } t \in [\kappa, T]\right\}, \text{ where } n \in \mathbb{N}$$

As a clarifying remark, we note that the collection of strategies $\mathcal{A}^{\mathrm{m},n}_\kappa$ in the definition above denotes the subclass of admissible Markovian hedging strategies which are *uniformly bounded* by n, where $n \in \mathbb{N}$. Plainly, by restricting the class of admissible strategies in this manner, we construct a variant of the best–response problem of investor i that is a standard stochastic optimal control problem, thereby allowing us to employ the convergence as well as continuity properties of the sequence $\left\{J^{i,\lambda,n}\right\}_{n \in \mathbb{N}}$ to prove the claims made in the statement of the lemma below. For a detailed proof the interested reader is directed to the technical appendix.

Lemma 5.1. Consider a deterministic initial time κ bounded above by T, along with a deterministic initial state vector $\left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}$ for investor i. Given an admissible trading strategy X^{-i} for investor $-i$, we have

(i) The best–response problem of investor i in the strategic hedging stochastic differential game is non–degenerate, that is,

$$\left|J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\right)\right| < \infty, \ \forall \lambda \in (-\infty, \infty)$$

(ii) The sequence $\left\{J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\right)\right\}_{n \in \mathbb{N}}$ of indirect utility functions converges,

$$\text{with } \lim_{n \to \infty} J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\right) = J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\right)$$

(iii) The indirect utility function $J^{i,\lambda,n}\big(\kappa,\,\cdot;\,X^{-i}\big)\,:\,\mathfrak{R}^{|Y^i|}\rightarrow(-\infty,0)$ is continuous for each $n\in\mathbb{N}$.

(iv) The indirect utility function $J^{i,\lambda}\big(\kappa,\,\cdot;\,X^{-i}\big)\,:\,\mathfrak{R}^{|Y^i|}\rightarrow(-\infty,0)$ is lower semi–continuous.

(v) The indirect utility function $J^{i,\lambda}\big(\kappa,\,\big[S_\kappa,\Delta^i_\kappa,\Delta^{-i}_\kappa,W^i_\kappa,W^{-i}_\kappa\big]^{\mathrm{T}};X^{-i}\big)\Big|_{\kappa=T}$ is lower semi–continuous with respect to its first argument.

Further, in the following ancillary lemma we establish a continuity result for an appropriately defined controlled state process associated with investor i's best–response problem which proves instrumental in establishing the invariance of the indirect utility function $J^{i,\lambda}$ with respect to the flow f^i. The claim proposed in the statement of the lemma mirrors that of (Gupta and Jacka, 2023, Lemma VI.2) and so does its proof, hence we omit the proof here.

Lemma 5.2. Consider a deterministic initial time κ bounded above by T, along with a decreasing sequence of deterministic times $\{\kappa_n\}_{n\in\mathbb{N}}\subseteq(\kappa,T]$, such that $\kappa_n\downarrow\kappa$. For a given a scalar $\varphi\in\mathfrak{R}$, define an admissible Markovian trading process $X^{\kappa_n}=\{x^{\kappa_n}_t\}_{t\in[0,T]}$, corresponding to the element κ_n of the sequence above, as follows

$$x^{\kappa_n}_t\;=\;\begin{cases}\varphi/\,(\kappa_n-\kappa)\,, & t\;\in\;[\kappa,\kappa_n]\\ 0, & t\in[0,\kappa)\bigcup(\kappa_n,T]\end{cases}\tag{5.1}$$

Given an admissible trading strategy X^{-i} of investor $-i$, suppose $Y^{i,\kappa_n}_{\kappa_n}$ denotes the time κ_n value of the controlled state process Y^{i,κ_n} defined as the solution of the stochastic differential equation (3.6), with $X^i=X^{\kappa_n}$ and a deterministic initial condition $Y^{i,\kappa_n}_\kappa=\big[S_\kappa,\Delta^i_\kappa,\Delta^{-i}_\kappa,W^i_\kappa,W^{-i}_\kappa\big]^{\mathrm{T}}$. The following convergence then holds $\mathbb{P}$–almost surely

$$\lim_{\kappa_n\downarrow\kappa}\,Y^{i,\kappa_n}_{\kappa_n}=f^i\big(\varphi,\,\big[S_\kappa,\Delta^i_\kappa,\Delta^{-i}_\kappa,W^i_\kappa,W^{-i}_\kappa\big]^{\mathrm{T}}\big)$$

The lemma below serves as a corollary to the previous lemma and documents another continuity result associated with the controlled state process of investor i's best–response problem in the strategic hedging game. As with the previous lemma, interested readers are directed to consult (Gupta and Jacka, 2023, Lemma VI.2, Lemma VI.3) for a proof of the result.

Lemma 5.3. Consider a deterministic initial time κ bounded above by T, along with a decreasing sequence of deterministic times $\{\kappa_n\}_{n\in\mathbb{N}}\subseteq(\kappa,T]$, such that $\kappa_n\downarrow\kappa$. Given a scalar $\varphi\in\mathfrak{R}$, consider an admissible Markovian trading process $\hat{X}^{\kappa_n}=\{\hat{x}^{\kappa_n}_t\}_{t\in[0,T]}$, associated with the element κ_n of the sequence above, defined as follows

$$x^{\kappa_n}_t\;=\;\begin{cases}-\varphi/\,(\kappa_n-\kappa)\,, & t\;\in\;[\kappa,\kappa_n]\\ 0, & t\in[0,\kappa)\bigcup(\kappa_n,T]\end{cases}\tag{5.2}$$

Suppose we are given an admissible trading strategy X^{-i} for investor $-i$ such that $\hat{Y}^{i,\kappa_n}_{\kappa_n}$ denotes the time κ_n value of the controlled state process $\hat{Y}^{i,\kappa_n}$ defined as the solution of the stochastic differential equation (3.6), with $X^i=\hat{X}^{\kappa_n}$ and deterministic initial condition $\hat{Y}^{i,\kappa_n}_\kappa=f^i\big(\varphi,\,\big[S_\kappa,\Delta^i_\kappa,\Delta^{-i}_\kappa,W^i_\kappa,W^{-i}_\kappa\big]^{\mathrm{T}}\big)$. The following convergence then holds $\mathbb{P}$–almost surely

$$\lim_{\kappa_n\downarrow\kappa}\,\hat{Y}^{i,\kappa_n}_{\kappa_n}=\big[S_\kappa,\Delta^i_\kappa,\Delta^{-i}_\kappa,W^i_\kappa,W^{-i}_\kappa\big]^{\mathrm{T}}$$

The following theorem provides a formal statement of the invariance result for the indirect utility functional $J^{i,\lambda}$ of investor i associated with the Black–Scholes–Cournot stochastic differential game. In addition to the apparatus laid out above, the proof of the invariance result makes use of an equicontinuity argument to tackle the issue of the utility function u^i being unbounded. We include a sketch of the proof in the technical appendix for the sake of completeness.

Theorem 5.1. Consider a deterministic initial time κ such that $\kappa < T$, as well as a deterministic initial state vector $\left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}$ associated with the best–response problem of investor i in the Black–Scholes–Cournot strategic hedging game. Given an arbitrary scalar $\varphi \in (-\infty, \infty)$ and an admissible strategy X^{-i} for investor $-i$, the indirect utility function $J^{i,\lambda}$ of investor i in the Black–Scholes–Cournot strategic hedging game is invariant with respect to the integral flow $f^i\big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\big)$, that is,

$$J^{i,\lambda}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) = J^{i,\lambda}\Big(\kappa, f^i\big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\big); X^{-i}\Big)$$

Proof. To begin with, we select a decreasing sequence of deterministic times $\{\kappa_n\} \in (\kappa, T]$ such that we have $\kappa_n \downarrow \kappa$, and corresponding to each element κ_n of the sequence, we associate a trading strategy X^{κ_n} for investor i defined as in (5.1). Since, we have $\mathcal{A}^{\mathrm{m},n}_\kappa \subseteq \mathcal{A}^{\mathrm{m}}_\kappa$ for all $n \in \mathbb{N}$, it follows that we can write

$$J^{i,\lambda}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) \geq J^{i,\lambda,n}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)$$

Further, we observe that X^{κ_n} as defined in (5.1) and X^{-i} are admissible trading strategies. It then follows from Definition 3.2, in conjunction with (Fleming and Soner, 2006, D.5, Appendix D) that for a fixed $n \in \mathbb{N}$ with $\kappa \leq s \leq \kappa_n$, the random variable Y^{i,κ_n}_s which denotes the time s value of the stochastic process Y^{i,κ_n} defined as in Lemma 5.2, converges to $\left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}$ $\mathbb{P}$–almost surely as $s \downarrow \kappa$. Moreover, in view of Lemma 5.1 (iii) we also have

$$J^{i,\lambda,n}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) \geq \limsup_{s \downarrow \kappa} J^{i,\lambda,n}\big(\kappa, Y^{i,\kappa_n}_s; X^{-i}\big)$$

Note that the choice of κ_n above was arbitrary which in conjunction with the fact that the inequality above remains valid when we pick s to be κ_n, leads us to the following relation by way of Lemma 5.1 (ii), Lemma 5.1 (iii), and Lemma 5.2

$$\begin{aligned}
J^{i,\lambda}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) &= \limsup_{n\to\infty} J^{i,\lambda,n}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) \\
&\geq \limsup_{n\to\infty} \Big(\limsup_{\kappa_n \downarrow \kappa} J^{i,\lambda,n}\big(\kappa, Y^{i,\kappa_n}_{\kappa_n}; X^{-i}\big)\Big) \\
&\geq \limsup_{n\to\infty} J^{i,\lambda,n}\Big(\kappa, f^i\Big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\Big); X^{-i}\Big) \\
&= J^{i,\lambda}\Big(\kappa, f^i\Big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\Big); X^{-i}\Big)
\end{aligned}$$

In order to establish the claim, we need to show that the inequality above holds with equality. To this end, we select a decreasing sequence of deterministic times $\{\kappa_n\} \in (\kappa, T]$ such that we have $\kappa_n \downarrow \kappa$. Further, corresponding to each element κ_n of the this sequence, we associate a trading strategy $\hat{X}^{\kappa_n}$ for investor i defined as in (5.2). Moreover, given that $\mathcal{A}^{\mathrm{m},n}_\kappa \subseteq \mathcal{A}^{\mathrm{m}}_\kappa$ for all $n \in \mathbb{N}$, we have

$$J^{i,\lambda}\Big(\kappa, f^i\Big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\Big); X^{-i}\Big) \geq J^{i,\lambda,n}\Big(\kappa, f^i\Big(\varphi, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\Big); X^{-i}\Big)$$

Given that the trading strategy $\hat{X}^{\epsilon_n}$ is admissible by construction and the trading strategy X^{-i} is admissible by hypothesis, it again follows from Definition 3.2, along with (Fleming and Soner, 2006, D.5, Appendix D) that for a fixed $n \in \mathbb{N}$ and $\kappa \leq s \leq \kappa_n$, the random variable $\hat{Y}_s^{i,\epsilon_n}$ which denotes time s value of the stochastic process $\hat{Y}^{i,\kappa_n}$ defined as in Lemma 5.3, converges to $f^i\left(\varphi, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}\right)$ $\mathbb{P}$-almost surely as $s \downarrow \kappa$. In view of Lemma 5.1 (iii), we then have

$$J^{i,\lambda,n}\left(\kappa, f^i\left(\varphi, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}\right); X^{-i}\right) \geq \limsup_{s \downarrow \kappa} J^{i,\lambda,n}\left(\kappa, \hat{Y}_s^{i,\kappa_n}; X^{-i}\right)$$

Since the choice of κ_n above was arbitrary and the inequality above remains valid in particular when we select s as κ_n, we obtain the following relation by way of Lemma 5.1 (ii), Lemma 5.1 (iii), as well as Lemma 5.3

$$\begin{aligned} J^{i,\lambda}\left(\kappa, f^i\left(\varphi, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}\right); X^{-i}\right) &= \limsup_{n \to \infty} J^{i,\lambda,n}\left(\kappa, f^i\left(\varphi, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}\right); X^{-i}\right) \\ &\geq \limsup_{n \to \infty} \left(\limsup_{\kappa_n \downarrow \kappa} J^{i,\lambda,n}\left(\kappa, \hat{Y}_{\kappa_n}^{i,\kappa_n}; X^{-i}\right)\right) \\ &\geq \limsup_{n \to \infty} J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) \\ &= J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) \end{aligned}$$

The claim in the statement of the theorem is then immediate from the above. □

5.2 Auxiliary Hedging Problem Invariance

We introduce some tools from the theory of piecewise constant controls, which shall be called upon in proving the invariance of the auxiliary hedging problem, corresponding to investor i's best–response problem, with respect to the flow f^i in the strategic hedging game.[24] To begin with, we consider a deterministic initial time κ bounded above by T, and suppose that $T_n = \left\{\kappa = t_0, t_1, \ldots, t_n = T\right\}$ denotes a partition of the time interval $[\kappa, T]$. For a given partition T_n, we define the diameter of T_n as $\max\left(t_{i+1} - t_i\right)$, with $0 \leq i \leq n-1$. Next, we select a countable subset $\mathbb{D} = \{\vartheta_k, k \in \mathbb{N}\} \subset \mathcal{D}$, where given that the state space $\mathcal{D}$ for the auxiliary control process is chosen to be a compact subset of the real line $\mathbb{R}$, we can assume that $\mathbb{D}$ is dense everywhere in $\mathcal{D}$ without loss of generality. Further, for $N \in \mathbb{N}$ fixed, we define a finite subset $\mathbb{D}_N$ of $\mathbb{D}$ as $\mathbb{D}_N = \{\vartheta_k\}, 1 \leq k \leq N$. Given a partition T_n, we first define the class of $\mathbb{D}_N$ valued piecewise constant controls, denoted as $\mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n, \mathbb{D}_N\right)$, by specifying that an adapted process $\mathcal{V} \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n, \mathbb{D}_N\right)$ if

(i) $\vartheta_t(\omega) \in \mathbb{D}_N, \forall (\omega, t) \in \Omega \times [\kappa, T]$

(ii) $\vartheta_t(\omega) = \vartheta_{t_{k+1}}(\omega), \forall (\omega, t) \in \Omega \times (t_k, t_{k+1}]$, with $k = 0, \ldots, n-1$.

We can successively generalize the definition of $\mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n, \mathbb{D}_N\right)$ to define $\mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n\right)$ which denotes the class of piecewise constant controls corresponding to the partition T_n, as well as the class of piecewise constant controls $\mathcal{A}_\kappa^{\mathrm{m},a,pc}$ as follows

$$\mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n\right) = \bigcup_{N \in \mathbb{N}} \mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n, \mathbb{D}_N\right)$$

$$\mathcal{A}_\kappa^{\mathrm{m},a,pc} = \bigcup_{T_n} \mathcal{A}_\kappa^{\mathrm{m},a,pc}\left(T_n\right)$$

[24] The treatment here is based on the classical treatise by Krylov (2008) and interested readers may consult the same for a detailed overview.

Additionally, we equip the class of admissible auxiliary strategies with a suitable metric, thereby formalizing the notion of *distance* between a pair of admissible strategies, while also defining a suitable topology for this class. To this end, let ρ denote a metric defined on the class of admissible auxiliary strategies $\mathcal{A}^{\mathrm{m},a}$ such that given $\mathcal{V}_1 = \left\{\vartheta_{1,s}\right\}_{s\in[\kappa,T]}$ and $\mathcal{V}_2 = \left\{\vartheta_{2,s}\right\}_{s\in[\kappa,T]} \in \mathcal{A}^{\mathrm{m},a}_{\kappa}$ we have

$$\rho\left(\mathcal{V}_1, \mathcal{V}_2\right) = \mathbb{E}\left[\int_{\kappa}^{T} |\,\vartheta_{1,s} - \vartheta_{2,s}\,|\, ds\right]$$

To verify that ρ is a metric, readers are advised to consult the proof of (Gupta and Jacka, 2023, Lemma VI.4). In subsequent discussion, a given sequence $\left\{\mathcal{V}_n\right\}_{n\in\mathbb{N}} \subseteq \mathcal{A}^{\mathrm{m},a}_{\kappa}$ of admissible auxiliary strategies will be understood to converge to an admissible auxiliary strategy $\mathcal{V}$, if and only if we have $\rho\left(\mathcal{V}_n, \mathcal{V}\right) \to 0$, as $n \to \infty$. Having assembled the essential ingredients from the theory of piecewise constant control, we recall a well known fact in stochastic optimal control theory in the following lemma, which states that an admissible strategy $\mathcal{V} \in \mathcal{A}^{\mathrm{m},a}_{\kappa}$ can be approximated by a sequence of piecewise constant strategies. For a proof, we direct the interested readers to see the proof of (Gupta and Jacka, 2023, Lemma VI.4).

Lemma 5.4. Given an admissible strategy $\mathcal{V}\in\mathcal{A}^{\mathrm{m},a}_{\kappa}$ and a sequence $\left\{T_n\right\}_{n\in\mathbb{N}}$ of nested partitions of the time interval $[\kappa, T]$ whose diameter converges to zero as $n \to \infty$, there exists a sequence of piecewise constant strategies $\left\{\mathcal{V}_n \in \mathcal{A}^{\mathrm{m},a,pc}_{\kappa}\left(T_n\right)\right\}_{n\in\mathbb{N}}$ which converges in the topology induced by the metric ρ to $\mathcal{V}$ as $n \to \infty$.

In view of the lemma above, given an admissible auxiliary strategy $\mathcal{V} \in \mathcal{A}^{\mathrm{m},a}_{\kappa}$ and a sequence $\left\{T_n\right\}_{n\in\mathbb{N}}$ of nested partitions of the time interval $[\kappa, T]$ whose diameter converges to zero as $n \to \infty$, we can find a sequence $\left\{\mathcal{V}_n \in \mathcal{A}^{\mathrm{m},a,pc}_{\kappa}\left(T_n\right)\right\}_{n\in\mathbb{N}}$ which converges to $\mathcal{V}$ as $n \to \infty$. Further, given an admissible hedging strategy $X^{-i} \in \mathcal{A}^{\mathrm{m}}_{\kappa}$ for investor $-i$, we consider the corresponding sequence of controlled auxiliary state processes $\left\{Z^{i,n}\right\}_{n\in\mathbb{N}}$ where recall that $Z^{i,n}$ is defined as the solution of the system of stochastic differential equations (4.10) with $\mathcal{V}^i = \mathcal{V}_n$. The following lemma establishes that the sequence of controlled auxiliary state processes $\left\{Z^{i,n}\right\}_{n\in\mathbb{N}}$ converges in an appropriate sense to the controlled auxiliary state process Z^i defined as the solution of the system of stochastic differential equations (4.10), with $\mathcal{V}^i = \mathcal{V}$. The proof of the lemma is identical to the proof (Gupta and Jacka, 2023, Lemma VI.5) and hence is omitted.

Lemma 5.5. Consider a deterministic initial time κ bounded above by T and a sequence $\left\{T_n\right\}_{n\in\mathbb{N}}$ of nested partitions of the time interval $[\kappa, T]$ whose diameter converges to zero as $n \to \infty$. Suppose $\left\{\mathcal{V}_n \in \mathcal{A}^{\mathrm{m},a,pc}_{\kappa}\left(T_n\right)\right\}_{n\in\mathbb{N}}$ denotes a sequence of piecewise constant auxiliary strategies which converges to the admissible auxiliary strategy $\mathcal{V} \in \mathcal{A}^{\mathrm{m},a}_{\kappa}$ in the limit as $n \to \infty$. Given the corresponding sequence of controlled auxiliary state processes $\left\{Z^{i,n}\right\}_{n\in\mathbb{N}}$ we can find a subsequence $\{n_m\}_{m\in\mathbb{N}}$, such that

$$\lim_{m\to\infty} \mathbb{E}\left[\sup_{s\in[\kappa,T]} \left|Z^{i,n_m}_s - Z^i_s\right|^2\right] = 0$$

In a similar vein, the following lemma formalizes the fact that we can approximate the indirect utility functional $F^{i,\lambda}$ corresponding to the auxiliary hedging problem of investor i through value functions of appropriately defined stochastic optimal control problems, where the collection of permissible auxiliary strategies is limited to the class of piecewise constant strategies, in the limit as the diameter of partition of the time interval $[\kappa, T]$ over which the piecewise constant control is defined becomes progressively smaller. For a detailed proof of the lemma, interested readers are directed to the technical appendix.

Lemma 5.6. Given a deterministic initial time κ bounded above by T, a deterministic initial state vector $\left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}$ for the auxiliary hedging problem associated with best–response problem of investor i in the strategic hedging game, and a sequence of nested partitions $\{T_n\}_{n\in\mathbb{N}}$ of the time interval $[\kappa, T]$ whose diameter converges to zero as $n \to \infty$ we have

$$\begin{aligned}\limsup_{n\to\infty} \sup_{\mathcal{V}^{i,n}\in\mathcal{A}^{\mathrm{m},a,pc}_\kappa(T_n)} \mathbb{E}\left[\sup_{\varphi\in\mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{\varphi}, \psi^{i,n}_T\right) + \lambda C \circ f^i\left(-\boldsymbol{\varphi}, \pi^{i,n}_T\right)\right)\right] \\ = F^{i,\lambda}\left(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)\end{aligned}$$

The theorem below provides a formal statement of the second invariance result connected with the invariance of the indirect utility functional $F^{i,\lambda}$ associated with the auxiliary hedging problem of investor i with respect to the integral flow f^i derived from the singular drift coefficient $\mathbf{b^i}$. The proof of the theorem essentially follows from the construction of the auxiliary hedging problem and makes heavy use of the framework of piecewise constant controls and the associated ancillary lemmas stated above.

Theorem 5.2. Consider a deterministic initial time κ bounded above by T, and a deterministic initial state vector $\left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}$ for the auxiliary hedging problem associated with best–response problem of investor i in the strategic hedging game. Given an arbitrary scalar $\varphi \in \mathcal{D}$ and an admissible strategy X^{-i} for investor $-i$, the indirect utility functional $F^{i,\lambda}$ corresponding to the auxiliary hedging problem of investor i is invariant with respect to the integral flow f^i, that is,

$$F^{i,\lambda}\left(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right) = F^{i,\lambda}\left(\kappa, f^i\left(\varphi, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}\right); X^{-i}\right)$$

Proof. We first consider the case when $\kappa = T$, and note that in the special instance when $\kappa = T$ the terminal value of the controlled auxiliary state Z^i_T is deterministic and equals Z^i_κ. From the definition of the indirect utility functional $F^{i,\lambda}$ we then have

$$\begin{aligned}F^{i,\lambda}\left(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)\Big|_{\kappa=T} &= \sup_{\mathrm{q}\in\mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{q}, \psi^i_T\right) + \lambda C \circ f^i\left(-\boldsymbol{q}, \pi^i_T\right)\right) \\ &= \sup_{\mathrm{q}\in\mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{q}, \psi^i_\kappa\right) + \lambda C \circ f^i\left(-\boldsymbol{q}, \pi^i_\kappa\right)\right)\end{aligned}$$

In view of the strict monotonicity of the utility function u^i, it is straightforward to check that the supremum on the right–hand side above satisfies translation invariance with respect to the flow f^i. This in turn implies that given an arbitrary scalar $\varphi \in \mathcal{D}$ we have

$$\begin{aligned}F^{i,\lambda}\left(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)\Big|_{\kappa=T} &= \sup_{\mathrm{q}\in\mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{q}+\varphi, \psi^i_T\right) + \lambda C \circ f^i\left(-\boldsymbol{q}+\varphi, \pi^i_T\right)\right) \\ &= \sup_{\mathrm{q}\in\mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{q}, f^i\left(\varphi, \psi^i_\kappa\right)\right) + \lambda C \circ f^i\left(-\boldsymbol{q}, f^i\left(\varphi, \pi^i_\kappa\right)\right)\right) \\ &= F^{i,\lambda}\left(\kappa, f^i\left(\varphi, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}\right); X^{-i}\right)\Big|_{\kappa=T}\end{aligned}$$

Next, we show that the claim holds true when $\kappa \in [0, T)$. To this end, we consider the deterministic strategy $\mathcal{V}^{i,0}$ defined as $\vartheta^{i,0}_t = 0$, for all $t \in [\kappa, T]$. It is straightforward to check that $\mathcal{V}^{i,0}$ is admissible in the sense of Definition 4.1. We let $Z^{i,0}_T$ denote the terminal value of the controlled auxiliary state process when investor i and investor $-i$ select admissible strategies $\mathcal{V}^{i,0}$ and X^{-i} respectively, with $Z^{i,0}_{\kappa} = \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}$. From the definition of $F^{i,\lambda}$ we then have

$$F^{i,\lambda}\Big(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\Big) \geq \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,0}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,0}_T\big)\big)\right]$$

Recalling the definition of controlled auxiliary state process, it follows that $Z^{i,0}_T = f^i\big(0, \hat{Y}^{i,0}_T\big)$ where $\hat{Y}^{i,0}$ denotes the abridged process defined as the solution of the multivariate stochastic differential equation (4.5), with initial condition given by $\hat{Y}^{i,0}_\kappa = f^i\big(-0, Z^{i,0}_\kappa\big)$. In view of this, the equation above can be rewritten as

$$F^{i,\lambda}\Big(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\Big) \geq \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \hat{Y}^{i,0,4}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \hat{Y}^{i,0,1}_T\big)\big)\right]$$

Note that $\hat{Y}^{i,0,1}_T$ and $\hat{Y}^{i,0,4}_T$ above represent the first and fourth component of the vector $\hat{Y}^{i,0}$. Further, given a partition T_n of the time interval $[\kappa, T]$, we consider a piecewise constant control $\mathcal{V}^{i,n} \in \mathcal{A}^{\mathrm{m},a,pc}_\kappa(T_n)$ such that $\vartheta^{i,n}_\kappa = \varphi \in \mathcal{D}$. Suppose $Z^{i,n}$ denotes the controlled auxiliary state process for investor i when investor i and investor $-i$ employ strategies $\mathcal{V}^{i,n}$ and X^{-i} respectively, with initial condition $Z^{i,n}_\kappa = f^i\big(\varphi, Z^{i,0}_\kappa\big)$. In view of the definition of controlled auxiliary state process it is straightforward to check that we have $Z^{i,n}_t = f^i\big(\vartheta^{i,n}_t, \hat{Y}^{i,0}_t\big)$ for $t \in [\kappa, T]$ which then gives us

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \hat{Y}^{i,0,4}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \hat{Y}^{i,0,1}_T\big)\big)\right] \\
&\quad = \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, f^i\left(-\vartheta^{i,n}_T, \psi^{i,n}_T\right)\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, f^i\left(-\vartheta^{i,n}_T, \pi^{i,n}_T\right)\big)\big)\right]
\end{aligned}$$

Observe that the supremum over the scalar $\mathbf{q}$ in the equation above satisfies translation invariance with respect to the flow f^i on account of strict monotonicity of the utility function u^i. Thus, by exploiting the time–shift property of the flow f^i in conjunction with Lemma 5.6 and recalling the fact that the choice of the partition T_n above was arbitrary, we arrive at the following

$$\begin{aligned}
&F^{i,\lambda}\Big(\kappa, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\Big) \\
&\quad \geq \limsup_{n\to\infty} \sup_{\mathcal{V}^{i,n}\in\mathcal{A}^{\mathrm{m},a,pc}_\kappa(T_n)} \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,n}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,n}_T\big)\big)\right] \\
&\quad = F^{i,\lambda}\Big(\kappa, f^i\Big(\varphi, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}\Big); X^{-i}\Big)
\end{aligned}$$

It remains to show that the inequality above holds with equality. To this end, we consider $\varphi \in \mathcal{D}$ along with the deterministic auxiliary strategy $\mathcal{V}^{i,\varphi}$ for investor i, with $\vartheta^{i,\varphi}_t = \varphi$, for $t \in [\kappa, T]$, where $\mathcal{V}^{i,\varphi} \in \mathcal{A}^{\mathrm{m},a}_\kappa$ in view of Definition 4.1. Suppose $Z^{i,\varphi}_T$ denotes the terminal value of the controlled auxiliary state process when investor i and investor $-i$ select strategies $\mathcal{V}^{i,\varphi}$ and X^{-i} respectively, with initial condition $Z^{i,\varphi}_\kappa = f^i\big(\varphi, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}\big)$. Recalling the definition of indirect utility function $F^{i,\lambda}$ we have

$$F^{i,\lambda}\Big(\kappa, f^i\Big(\varphi, \left[\pi^i_\kappa, \mu^i_\kappa, \mu^{-i}_\kappa, \psi^i_\kappa, \psi^{-i}_\kappa\right]^{\mathcal{T}}\Big); X^{-i}\Big) \geq \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,\varphi}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,\varphi}_T\big)\big)\right]$$

Further, from the construction of controlled auxiliary state process we have $Z^{i,\varphi}_T = f^i(\varphi, \hat{Y}^{i,0}_T)$ where $\hat{Y}^{i,0}$ denotes the abridged process defined as the solution of the stochastic differential equation

(4.5), with initial condition given by $\hat{Y}^{i,0}_{\kappa} = f^i(-\varphi, Z^{i,\varphi}_{\kappa}) = \left[\pi^i_{\kappa}, \mu^i_{\kappa}, \mu^{-i}_{\kappa}, \psi^i_{\kappa}, \psi^{-i}_{\kappa}\right]^{\mathcal{T}}$. In view of this, the right–hand side of the inequality above can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,\varphi}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,\varphi}_T\big)\big)\right] \\
&\quad = \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, f^i\big(\varphi, \hat{Y}^{i,0,4}_T\big)\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, f^i\big(\varphi, \hat{Y}^{i,0,1}_T\big)\big)\big)\right]
\end{aligned}$$

Next, we consider a partition T_j of the time interval $[\kappa, T]$, as well as an admissible piecewise constant auxiliary strategy $\mathcal{V}^{i,j} \in \mathcal{A}^{\mathrm{m},a,pc}_{\kappa}\big(T_j\big)$ such that $\vartheta^{i,j}_{\kappa} = 0 \in \mathcal{D}$. We let $Z^{i,j}$ denote the controlled auxiliary state process when investor i and investor $-i$ select strategies $\mathcal{V}^{i,j}$ and X^{-i} respectively. Following the definition of controlled auxiliary state process we have $Z^{i,j}_{\kappa} = f^i(0, \hat{Y}^{i,0}_{\kappa}) = \left[\pi^i_{\kappa}, \mu^i_{\kappa}, \mu^{-i}_{\kappa}, \psi^i_{\kappa}, \psi^{-i}_{\kappa}\right]^{\mathcal{T}}$. In view of translation invariance of the supremum in the equation above as well as the time–shift property of f^i we then have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, f^i\big(\varphi, \hat{Y}^{i,0,4}_T\big)\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, f^i\big(\varphi, \hat{Y}^{i,0,1}_T\big)\big)\big)\right] \\
&\quad = \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,j}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,j}_T\big)\big)\right]
\end{aligned}$$

Once again we appeal to the time–shift property of f^i, the translation invariance of the supremum over the scalar $\mathbf{q}$ in the equation above which follows on account of the strict monotonicity of the utility function u^i, as well as Lemma 5.6 to arrive at

$$\begin{aligned}
&F^{i,\lambda}\Big(\kappa, \left[\pi^i_{\kappa}, \mu^i_{\kappa}, \mu^{-i}_{\kappa}, \psi^i_{\kappa}, \psi^{-i}_{\kappa}\right]^{\mathcal{T}}; X^{-i}\Big) \\
&\quad = \limsup_{j\to\infty} \sup_{\mathcal{V}^{i,j}\in\mathcal{A}^{\mathrm{m},a,pc}_{\kappa}(T_j)} \mathbb{E}\left[\sup_{\mathbf{q}\in\mathfrak{R}} u^i\big(f^i\big(-\boldsymbol{q}, \psi^{i,j}_T\big) + \lambda C \circ f^i\big(-\boldsymbol{q}, \pi^{i,j}_T\big)\big)\right] \\
&\quad \leq F^{i,\lambda}\Big(\kappa, f^i\Big(\varphi, \left[\pi^i_{\kappa}, \mu^i_{\kappa}, \mu^{-i}_{\kappa}, \psi^i_{\kappa}, \psi^{-i}_{\kappa}\right]^{\mathcal{T}}\Big); X^{-i}\Big)
\end{aligned}$$

As earlier, the equality above relies on the fact that the choice of the partition T_j above was arbitrary. The claim in the statement of the theorem is immediate from the above. □

5.3 Singular & Auxiliary Problem Equivalence

Theorem 5.3 below states the equivalence result for the singular best–response hedging problem and the auxiliary hedging problem, which represents the principal technical contribution of the present work. Specifically, we establish that indirect utility functional of an investor corresponding to the best–response hedging problem coincides with the value function of the auxiliary hedging problem constructed in the previous section.

This equivalence result along with the fact that the auxiliary hedging problem is a standard stochastic optimal control problem allows us to resolve the issue of potential singularity of the best–response hedging problem of investor i by facilitating characterization of the associated indirect utility functional of investor i through standard techniques.

The proof of the theorem relies crucially on the assumption of Markovian strategies, due to which we are able to prove the desired equivalence by generalizing the proof of Lasry and Lions

(2000, Theorem 1) to a strategic setup, with primary as well as derivative financial assets. The proof proceeds by establishing that the indirect utility functional $J^{i,\lambda}$ associated with the best–response hedging problem serves as an upper bound for the value function $F^{i,\lambda}$ of the auxiliary hedging problem through fairly standard arguments involving the piecewise constant control framework and the construction of the auxiliary hedging problem.

To show that this upper bound is tight, we consider a variant of the best–response hedging problem with uniformly bounded controls, whose value function is denoted by $J^{i,\lambda,n}$. With the help of the invariance result for the auxiliary hedging problem established in Theorem 5.2, we show that $F^{i,\lambda}$ serves as an upper bound for $J^{i,\lambda,n}$ by appealing to the theory of viscosity solutions. The desired result then follows as a consequence of Lemma 5.1.

Theorem 5.3. Consider a deterministic initial time κ such that $\kappa < T$, as well as a deterministic initial state vector $\left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}$ for the best–response hedging problem of investor i in the strategic hedging game. Given an admissible strategy X^{-i} for investor $-i$, the indirect utility functional $J^{i,\lambda}$ of investor i in the strategic hedging game is equivalent to the value function $F^{i,\lambda}$ of the auxiliary hedging problem, that is,

$$J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right) = F^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)$$

Proof. To begin with, we show that the indirect utility functional of investor i associated with the best–response hedging problem serves as an upper bound for the value function of the auxiliary hedging problem, that is, we aim to show

$$F^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right) \leq J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathcal{T}}; X^{-i}\right)$$

Given admissible hedging strategy X^{-i} for investor $-i$, we consider the zero control $x^i_t = 0$, $t \in [\kappa, T]$ for investor i. It follows from the definition of the abridged process $\hat{Y}^i$ (4.5) that if the investors follow these respective strategies up to time $T-\epsilon$, where $\epsilon > 0$, then $Y^i_{T-\epsilon} = \hat{Y}^i_{T-\epsilon}$ with initial condition $\hat{Y}^i_\kappa = Y^i_\kappa$. Since the zero control is vacuously admissible, it is immediate from the definition of the indirect utility functional $J^{i,\lambda}$ that we have

$$J^{i,\lambda}\left(\kappa, Y^i_\kappa; X^{-i}\right) \geq J^{i,\lambda}\left(T-\epsilon, \hat{Y}^i_{T-\epsilon}; X^{-i}\right)$$

Recalling the invariance result for the singular best–response hedging problem established in Theorem 5.1, in conjunction with the fact that the choice of ϵ above was arbitrary, leads us to the following equation given $\varphi \in \mathbb{R}$

$$\begin{aligned} J^{i,\lambda}\left(\kappa, Y^i_\kappa; X^{-i}\right) &\geq \sup_{s \in [T-\epsilon, T)} J^{i,\lambda}\left(s, f^i\left(\varphi, \hat{Y}^i_s\right); X^{-i}\right) \\ &\geq \lim_{\epsilon \downarrow 0} \inf_{s \in [T-\epsilon, T)} J^{i,\lambda}\left(s, f^i\left(\varphi, \hat{Y}^i_s\right); X^{-i}\right) \end{aligned}$$

We know from Lemma 5.1 (iv), (v) that $J^{i,\lambda}$ is lower semi–continuous with respect to its second argument, and lower semi–continuous in its first argument at terminal time T respectively, which in conjunction with the continuity of the integral flow f^i and the fact that the choice of φ was arbitrary then implies that we have

$$\lim_{\epsilon \downarrow 0} \inf_{s \in [T-\epsilon, T)} J^{i,\lambda}\left(s, f^i\left(\varphi, \hat{Y}^i_s\right); X^{-i}\right) \geq u^i\left(f^i\left(\varphi, \mathrm{W}^i_T\right) + \lambda C \circ f^i\left(\varphi, S_T\right)\right)$$

$$\sup_{\varphi \in \mathbb{R}} \lim_{\epsilon \downarrow 0} \inf_{s \in [T-\epsilon, T)} J^{i,\lambda}\left(s, f^i\left(\varphi, \hat{Y}^i_s\right); X^{-i}\right) \geq \sup_{\varphi \in \mathbb{R}} u^i\left(f^i\left(\varphi, \widehat{\mathrm{W}}^i_T\right) + \lambda C \circ f^i\left(\varphi, \hat{S}_T\right)\right)$$

Note that on the right–hand above we have used $\hat{Y}_T^i = \left[\hat{S}_T, \hat{\Delta}_T^i, \hat{\Delta}_T^{-i}, \widehat{W}_T^i, \widehat{W}_T^{-i}\right]^{\mathcal{T}}$. Further, let T_n denote an arbitrary partition of the time interval $[\kappa, T]$, $\mathcal{V}^i \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)$ an admissible piecewise constant strategy with $\vartheta_\kappa^i = 0$, and $Z^{i,\mathcal{V}^i}$ the auxiliary state process when investor i and investor $-i$ employ strategies $\mathcal{V}^i$ and X^{-i} respectively. Recalling the definition of the auxiliary state process, we see that the initial controlled auxiliary state value equals $Z_\kappa^{i,\mathcal{V}^i} = f^i\big(0, \hat{Y}_\kappa^i\big) = Y_\kappa^i$, and that the terminal controlled auxiliary state value is given by $Z_T^{i,\mathcal{V}^i} = f^i\big(\vartheta_T^i, \hat{Y}_T^i\big)$ by way of which we have

$$\sup_{\varphi \in \mathbb{R}} \lim_{\epsilon \downarrow 0} \inf_{s \in [T-\epsilon, T)} J^{i,\lambda}\big(s, f^i\big(\varphi, \hat{Y}_s^i\big); X^-\big) \geq \sup_{\varphi \in \mathbb{R}} u^i\Big(f^i\big(\varphi, f^i\big(-\vartheta_T^i, Z_T^{i,\mathcal{V}^i}\big)\big) + \lambda C \circ f^i\big(\varphi, f^i\big(-\vartheta_T^i, Z_T^{i,\mathcal{V}^i}\big)\big)\Big)$$

Further, by way of the translation invariance of the supremum in the equation above, which follows on account of the flow property of f^i along with the strict monotonicity of the utility function u^i, as well as the arbitrary choice of the partition T_n above we have

$$J^{i,\lambda}\big(\kappa, Y_\kappa^i; X^{-i}\big) \geq \sup_{\mathcal{V} \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)} \mathbb{E}\left[\sup_{\varphi \in \mathbb{R}} u^i\Big(f^i\big(-\varphi, Z_T^{i,\mathcal{V}^i}\big) + \lambda C \circ f^i\big(-\varphi, Z_T^{i,\mathcal{V}^i}\big)\Big)\right]$$

Given the equation above, we consider the limit as $n \to \infty$ and invoke Lemma 5.6 from which it is immediate that we have

$$F^{i,\lambda}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big) \leq J^{i,\lambda}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big)$$

In order to prove the claim, it remains to show that the upper bound is tight in the sense that the inequality holds with equality. To this end, we show that the reverse inequality holds true, that is,

$$J^{i,\lambda}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big) \leq F^{i,\lambda}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big)$$

Recall that by definition value function of the auxiliary hedging problem $F^{i,\lambda}$ is a viscosity supersolution of the associated Hamilton–Jacobi–Bellman equation. If $\underline{F}^{i,\lambda}$ denotes the lower semi–continuous envelope of $F^{i,\lambda}$, that is, the largest lower semi–continuous minorant of $F^{i,\lambda}$, whose existence follows from the fact that $F^{i,\lambda}$ is bounded below, then given $g \in \mathbb{C}^{1,2}\big((0,T] \times \mathbb{R}^{|Z^i|}; \mathbb{R}\big)$ such that $\underline{F}^{i,\lambda} - g$ attains a global minimum at $(\kappa^*, z^*) \in (0,T) \times \mathbb{R}^{|Z^i|}$, we have

$$-D_1 g(\kappa^*, z^*) + \sup_{\varphi \in \mathcal{D}} \left\{ \big\langle -D_2 g(\kappa^*, z^*), \beta^i(\varphi, z^*)\, x^{-i} \big\rangle - \frac{1}{2} \mathrm{tr}\left(D_2^2 g(\kappa^*, z^*)\, \chi^i(\varphi, z^*)\, \chi^i(\varphi, z^*)^{\mathcal{T}}\right) \right\} \geq 0$$

Since, $\mathcal{D}$ is assumed to be a compact subset of $\mathbb{R}$ and given that the functions β^i and χ^i are continuous in their first argument, we can find a maximizer $\varphi^* \in \mathcal{D}$ such that the equation above leads us to

$$-D_1 g(\kappa^*, z^*) - \big\langle D_2 g(\kappa^*, z^*), \beta^i(\varphi^*, z^*)\, x^{-i} \big\rangle - \frac{1}{2} \mathrm{tr}\left(D_2^2 g(\kappa^*, z^*)\, \chi^i(\varphi^*, z^*)\, \chi^i(\varphi^*, z^*)^{\mathcal{T}}\right) \geq 0$$

Moreover, recalling that the auxiliary state value z^*, and the abridged process value $\hat{y}$ defined in (4.5) are related as $z^* = f^i(\varphi^*, \hat{y}^*)$ we can rewrite the equation above as

$$\begin{aligned} -D_1 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big) &- \big\langle D_2 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big), \beta^i\big(\varphi^*, f^i(\varphi^*, \hat{y}^*)\big)\, x^{-i} \big\rangle \\ &- \frac{1}{2} \mathrm{tr}\left(D_2^2 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big)\, \chi^i\big(\varphi^*, f^i(\varphi^*, \hat{y}^*)\big)\, \chi^i\big(\varphi^*, f^i(\varphi^*, \hat{y}^*)\big)^{\mathcal{T}}\right) \geq 0 \end{aligned}$$

It is straightforward to check that the definition of β^i and χ^i as outlined in (4.9) implies that the equation above can be rewritten as

$$-D_1 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big) - \Big\langle D_2 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big)\,, D_2 f^i(\varphi^*, \hat{y}^*)\, a^i(\hat{y}^*)\, x^{-i}\Big\rangle$$
$$- \frac{1}{2}\mathrm{tr}\left(D_2 f^i(\varphi^*, \hat{y}^*)^{\mathcal{T}}\, D_2^2 g\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big)\, D_2 f^i(\varphi^*, \hat{y}^*)\, \mathrm{v}^i(\hat{y}^*)\, \mathrm{v}^i(\hat{y}^*)^{\mathcal{T}}\right) \geq 0 \quad (5.3)$$

Further, we define $\hat{g} \in \mathbb{C}^{1,2}\big((0,T] \times \mathbb{R}^{|Z^i|}; \mathbb{R}\big)$ as $\hat{g}(\kappa, \hat{y}) = g\big(\kappa, f^i(\varphi^*, \hat{y})\big)$, from which we note the following

$$\begin{aligned}
D_1 \hat{g}(\kappa, \hat{y}) &= D_1 g\big(\kappa, f^i(\varphi^*, \hat{y})\big) \\
D_2 \hat{g}(\kappa, \hat{y}) &= D_2 g\big(\kappa, f^i(\varphi^*, \hat{y})\big)\, D_2 f^i(\varphi^*, \hat{y}) \\
D_2^2\, \hat{g}(\kappa, \hat{y}) &= D_2 f^i(\varphi^*, \hat{y})^{\mathcal{T}} D_2^2\, g\big(\kappa, f^i(\varphi^*, \hat{y})\big)\, D_2 f^i(\varphi^*, \hat{y}) + D_2 g\big(\kappa, f^i(\varphi^*, \hat{y})\big)\, D_2^2\, f^i(\varphi^*, \hat{y}) \\
&= D_2 f^i(\varphi^*, \hat{y})^{\mathcal{T}} D_2^2\, g\big(\kappa, f^i(\varphi^*, \hat{y})\big)\, D_2 f^i(\varphi^*, \hat{y})
\end{aligned}$$

The last equality above is a consequence of the fact that we have $D_2^2 f^i(\varphi, \hat{y}) = \mathbf{0}_{5\times 5}$ for all $(\varphi, \hat{y})$. From the definition of $\hat{g}$ and the invariance result for the auxiliary hedging problem established in Theorem 5.2, it follows that when $\underline{F}^{i,\lambda} - g$ attains a global minimum at $\big(\kappa^*, f^i(\varphi^*, \hat{y}^*)\big) \in (0,T) \times \mathbb{R}^{|Z^i|}$, $\underline{F}^{i,\lambda}\big(\,\cdot\,, f^i(\varphi^*, \cdot\,)\,; X^{-i}\big) - \hat{g}$ attains a global minimum at $(\kappa^*, \hat{y}^*)$. Thus, (5.3) can be rewritten as

$$-D_1 \hat{g}\,(\kappa^*, \hat{y}^*) - \Big\langle D_2 \hat{g}\,(\kappa^*, \hat{y}^*)\,, a^i(\hat{y}^*)\, x^{-i}\Big\rangle - \frac{1}{2}\mathrm{tr}\left(D_2^2 \hat{g}\,(\kappa^*, \hat{y}^*)\, \mathrm{v}^i(\hat{y}^*)\, \mathrm{v}^i(\hat{y}^*)^{\mathcal{T}}\right) \geq 0$$

It follows from preceding arguments that $F^{i,\lambda}\big(\kappa, f^i(\varphi, \hat{y}^i)\,; X^{-i}\big)$ serves as a viscosity supersolution to the following second–order partial differential equation in ψ where $n \in \mathbb{N}$ fixed

$$-D_1 \psi(\kappa, y) + n\left|\Big\langle D_2 \psi(\kappa, y)\,, b^i(y)\Big\rangle\right| - \Big\langle D_2 \psi(\kappa, y)\,, a^i(y)\, x^{-i}\Big\rangle - \frac{1}{2}\mathrm{tr}\left(D_2^2\, \psi(\kappa, y)\, \mathrm{v}^i(y)\, \mathrm{v}^i(y)^{\mathcal{T}}\right) = 0 \quad (5.4)$$

Note that the terminal condition for the equation above equals

$$F^{i,\lambda}\Big(T, \big[S_T, \Delta_T^i, \Delta_T^{-i}, W_T^i, W_T^{-i}\big]^{\mathrm{T}}; X^{-i}\Big) = \sup_{\varphi \in \mathbb{R}} u^i\big(f^i\big(-\varphi, \psi_T^i\big) + \lambda C \circ f^i\big(-\varphi, \pi_T^i\big)\big) \geq u^i\big(\mathrm{W}_T^i + \lambda C \circ S_T\big)$$

Further, we consider $J^{i,\lambda,n}$ which denotes indirect utility of ith investor's best–response hedging problem when hedging process X^i is constrained to lie in $\mathcal{A}_\kappa^{\mathrm{m},n}$. By definition, $J^{i,\lambda,n}$ satisfies the viscosity subsolution property for the associated Hamilton–Jacobi–Bellman equation. Letting $\overline{J}^{\,i,\lambda,n}$ denote the upper semi–continuous envelope of $J^{i,\lambda,n}$, that is, the smallest upper semi–continuous majorant of $J^{i,\lambda}$, and given $h \in \mathbb{C}^{1,2}\big((0,T] \times \mathbb{R}^{|Y^i|}; \mathbb{R}\big)$ such that $\overline{J}^{\,i,\lambda,n} - h$ attains a global maximum at $(\kappa^*, y^*) \in (0,T) \times \mathbb{R}^{|Y^i|}$, we have

$$-D_1 h\,(\kappa, y) + \sup_{x \in \overline{\mathcal{B}_n(0)}} \Big\{\Big\langle -D_2 h\,(\kappa, y)\,, b^i(y)\, x\Big\rangle\Big\} - \Big\langle D_2 h\,(\kappa, y)\,, a^i(y)\, x^{-i}\Big\rangle$$
$$- \frac{1}{2}\mathrm{tr}\left(D_2^2\, h\,(\kappa, y)\, \mathrm{v}^i(y)\, \mathrm{v}^i(y)^{\mathcal{T}}\right) \leq 0$$

It follows from the preceding arguments that $J^{i,\lambda,n}$ serves as a viscosity subsolution to (5.4), with terminal condition given by

$$J^{i,\lambda,n}\big(T, y; X^{-i}\big) = u^i\big(\mathrm{W}_T^i + \lambda C \circ S_T\big)$$

In view of the comparison principle for viscosity solutions of second–order partial differential equations (Fleming and Soner, 2006, Theorem V.9.1) and recalling that $z = f^i(\varphi^*, \hat{y})$, we obtain

$$J^{i,\lambda,n}\left(\kappa, y^i; X^{-i}\right) \leq F^{i,\lambda}\left(\kappa, f^i(\varphi, \hat{y}^i); X^{-i}\right) = F^{i,\lambda}\left(\kappa, z^i; X^{-i}\right)$$

By way of Lemma 5.1 it follows from the equation above that we have

$$J^{i,\lambda}\left(\kappa, y^i; X^{-i}\right) \leq F^{i,\lambda}\left(\kappa, z^i; X^{-i}\right)$$

The claim in the statement of the theorem follows immediately from the above. □

6. Strategic Indifference Price

In this section, we compute the investor specific utility indifference price of a derivative. As stated earlier, this requires solving for the Markov–Nash equilibrium payoffs of a pair of singular stochastic differential games. We start by solving for the value functions (3.7) and (3.8) corresponding to each investor's best–response problem and, subsequently invoking (3.9) which provides an implicit representation of the strategic indifference price. In general, the computation of Markov–Nash equilibrium payoffs and the strategic indifference price must be done numerically.

In order to compute the best–response value functions of the investors, we rely on the equivalence result proved above in conjunction with the fact that the auxiliary hedging problem is a standard control problem. Given the assumption of Markovian strategies, we are able to characterize the value functions as viscosity solutions of coupled HJB equations via a primal approach. To solve for Markov–Nash equilibrium payoffs and the resulting indifference price, we employ a numerical approach based on the finite difference method for a system of coupled second–order fully nonlinear partial differential equations in conjunction with the standard value function iteration method.

6.1 Best–Response Value Functions

It is relatively straightforward to characterize the value function defined by (3.7) using the Hamilton–Jacobi–Bellman approach based on the dynamic programming principle. We follow Fleming and Soner (2006, IV.10) and Krylov (2008), which prove that the desired value function must necessarily be a solution to (6.1) in a generalized sense.

$$\begin{aligned} &D_1F^{-i,0}\left(s, Z_s^{-i}; X^i\right) + \sup_{\nu \in \Delta}\Big\{ \left\langle D_2F^{-i,0}\left(s, Z_s^{-i}; X^i\right), \beta^{-i}\left(\nu, Z_s^{-i}\right)\right\rangle \\ &+ \frac{1}{2}\operatorname{tr}\left(D_2^2F^{-i,0}\left(s, Z_s^{-i}; X^i\right)\chi^{-i}\left(\nu, Z_s^{-i}\right)\chi^{-i}\left(\nu, Z_s^{-i}\right)^{\mathcal{T}}\right)\Big\} = 0 \\ &\text{with terminal condition } F^{-i,0}\left(T, Z_T^{-i}; X^i\right) = u^{-i}\left(\psi_T^{-i} + \frac{(\pi_T^{-i})^2}{2\theta^{-i}}\right) \end{aligned} \tag{6.1}$$

Similarly, in order to obtain a characterization for the value function defined by (3.8) using the Hamilton–Jacobi–Bellman approach, we again rely on the dynamic programming approach followed in Fleming and Soner (2006, IV.10) and Krylov (2008) to claim that the desired value function should solve (6.2) in a generalized or viscosity sense for $0 < s < T$

$$\begin{aligned} &D_1F^{i,\lambda}\left(s, Z_s^i; X^{-i}\right) + \sup_{\nu \in \Delta}\Big\{ \left\langle D_2F^{i,\lambda}\left(s, Z_s^i; X^{-i}\right), \beta^i\left(\nu, Z_s^i\right)\right\rangle \\ &+ \frac{1}{2}\operatorname{tr}\left(D_2^2F^{i,\lambda}\left(s, Z_s^i; X^{-i}\right)\chi^i\left(\nu, Z_s^i\right)\chi^i\left(\nu, Z_s^i\right)^{\mathcal{T}}\right)\Big\} = 0 \end{aligned} \tag{6.2}$$

In order to establish the sufficiency of a (viscosity) solution of (6.1) and (6.2) to be the required value functions of the corresponding auxiliary hedging problems one can invoke a routine verification lemma, see Pham (2009). To complete the characterization of the value functions, it remains to fix a terminal condition for the value function defined by (3.8). To this end, we recall the definition of the auxiliary hedging problem and note that the terminal condition is defined as

$$F^{i,\lambda}\left(T, Z_T^i; X^{-i}\right) = \sup_{\boldsymbol{\varphi} \in \mathfrak{R}} u^i\left(f^i\left(-\boldsymbol{\varphi}, \psi_T^i\right) + \lambda C \circ f^i\left(-\boldsymbol{\varphi}, \pi_T^i\right)\right)$$

The derivation of the terminal condition in this instance is slightly involved since the derivative holding λ is assumed to be non–zero in general, and we consider a setup which is general enough to consider market manipulation through derivative payoffs. Thus, we recall Assumption 5.2 in conjunction with Dini's implicit function theorem Dontchev and Rockafellar (2009, Chapter 1, Page 5) to deduce that there exists a function $\hat{f}\left(\mathrm{v}^i, \pi^i, \lambda, \theta^i\right) \in \mathbb{C}^1$ such that the terminal condition above can be re–written as

$$F^{i,\lambda}\left(T, Z_T^i; X^{-i}\right) = u^i\left(\psi_T^i + \frac{(\pi_T^i)^2}{2\theta^i} + \lambda C \circ \hat{f}^i\left(\mathrm{v}_T^i, \pi_T^i, \lambda, \theta^i\right) - \frac{\theta^i}{2}\left(\hat{f}^i\left(\mathrm{v}_T^i, \pi_T^i, \lambda, \theta^i\right)\right)^2\right) \tag{6.3}$$

A remark is in order concerning the difference in the terminal conditions specified by (6.1) and (6.3). This difference stems from the fact that a large investor with a nonzero derivative position faces an additional trade–off when compared with a large investor trading exclusively in the primary assets. With a nonzero derivative position, an investor can exploit the price impact on the underlying primary asset to manipulate the derivative payoff in a favorable direction, albeit incurring additional liquidity costs on account of its price impact. This phenomenon bears marked resemblance to *punching the close* in derivative markets defined in earlier literature, see Kumar and Seppi (1992) and Horst and Naujokat (2011).

The investor thus opts to move to a point on the flow orbit where these two effects are exactly counterbalanced. That is, the marginal gain from manipulating the derivative payoff exactly equals the marginal liquidity cost arising due to nonzero price impact. This is precisely the effect captured by the last two terms which appear in the argument of the utility function in (6.3). It is worth mentioning that as a consequence of the implicit function theorem, it also follows that the value of $\hat{f}\left(\mathrm{v}^i, \pi^i, \lambda, \theta^i\right)$ when the investor chooses not to invest in derivative security, that is when $\lambda = 0$, equals zero and thus the terminal condition reduces to the familiar form when derivative holdings are zero.

6.2 Liquidity Adjusted Black–Scholes Equation

In general, one must employ numerical methods to compute the utility indifference price of a derivative when investors employ Markovian trading strategies. This is because the Hamilton–Jacobi–Bellman equation (6.2) associated with the auxiliary hedging problem when the derivative position λ is nonzero, admits no closed form analytical solution in general. In fact, excepting the case when there is a single large investor, it is not possible to derive a closed form analytical solution even for (6.1), which represents the Hamilton–Jacobi–Bellman equation associated with the auxiliary hedging problem when λ equals zero.

Nevertheless, with the aim of aiding intuition, we defer our discussion of the general numerical approach to later, and focus attention on the special case of determining investor *i*'s indifference price when investor $-i$ is constrained to follow a deterministic strategy. Besides tractability, this assumption is motivated by earlier literature on strategic trading in financial markets.

For example, the seminal paper by Almgren and Chriss (2001) on optimal execution with price impact assumes that over shorter time horizon, or alternatively while considering high–frequency trading, the aggregate market volume faced by an investor is deterministic. A similar justifying argument is made for the consideration of deterministic strategies in Guéant and Pu (2017), while Gârleanu (2009) assume that continuous readjustments to trading strategies are infeasible.

Under this simplifying assumption, we are able to solve for the value functions which solve the Hamilton–Jacobi–Bellman equations (6.1) and (6.2) respectively. In fact, under the assumption that investor $-i$ follows a deterministic strategy, the resulting problem resembles the classical portfolio choice problem in Merton (1973) which simplifies the proof considerably. In fact, the proof of the following proposition follows directly from the arguments in (Lions and Lasry, 2007, Theorem 2.1) and (Musiela and Zariphopoulou, 2004, Theorem 2), and hence we direct the interested reader there for a detailed proof.

Proposition 6.1. Suppose $X^{-i,*}$ denotes the deterministic Markov–Nash equilibrium strategy of investor $-i$ in the strategic hedging and let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ such that $B_t - \theta^{-i}\int_\kappa^t x_s^{-i,*}ds$ is a $\mathbb{Q}$–Brownian motion. The best–response value function of investor i with derivative position λ is then given by

$$\begin{aligned} &J^{i,\lambda}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, \mathrm{W}_\kappa^i, \mathrm{W}_\kappa^{-i}\big]^{\mathcal{T}}; X^{-i}\Big) \\ &= -\frac{1}{\gamma^i}\exp\left\{-\gamma^i\left(W_\kappa^i + \frac{S_\kappa^2}{2\theta^i} - \frac{1}{2\sigma^2}\int_\kappa^T \theta^{-i}x_t^{-i,*}\,dt - (T-\kappa)\frac{\sigma^2}{2\theta^i}\right)\right\}\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\gamma^i\lambda C\circ\hat{f}\left(S_T\right)\right\}\right] \end{aligned}$$

We make two remarks in relation to the proposition above. First, note that under the probability measure $\mathbb{Q}$, the *auxiliary* price π^i is a martingale. Moreover, the Radon–Nikodym derivative of $\mathbb{Q}$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{\int_\kappa^T \theta^{-i}x_s^{-i,*}dB_s + \frac{(\theta^{-i})^2}{2}\int_\kappa^T \left(x_s^{-i,*}\right)^2 ds\right\}$$

Henceforth, we refer to $\mathbb{Q}$ as Markov–Nash pricing measure. Second, it is immediate from the proposition above that the best–response value function of investor i with zero derivative holding is given by

$$\begin{aligned} &J^{i,0}\Big(\kappa, \big[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, \mathrm{W}_\kappa^i, \mathrm{W}_\kappa^{-i}\big]^{\mathcal{T}}; X^{-i}\Big) \\ &= -\frac{1}{\gamma^i}\exp\left\{-\gamma^i\left(W_\kappa^i + \frac{S_\kappa^2}{2\theta^i} - \frac{1}{2\sigma^2}\int_\kappa^T \theta^{-i}x_t^{-i,*}\,dt - (T-\kappa)\frac{\sigma^2}{2\theta^i}\right)\right\} \end{aligned}$$

In view of the above, we can compute the indifference (bid) price of a derivative using the definition outlined in Section 3 in a straightforward manner. The following proposition gives a semi-closed form characterization of the indifference bid price for investor i, where the proof is immediate in view of (Musiela and Zariphopoulou, 2004, Theorem 3).

Proposition 6.2. Let $\mathbb{Q}$ denote the Markov–Nash pricing measure and $\kappa \le T$ ne a given deterministic initial time. The indifference bid price for investor i for λ units of the derivative with payoff function C is characterized as

$$C_\kappa^{B,i}(\lambda) = \frac{1}{\gamma^i}\log\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{\gamma^i\lambda C\circ\hat{f}\left(S_T\right)\right\}\right]$$

Some remarks are in order here regarding the semi–closed form pricing characterization obtained above. In contrast to the case of complete markets, the strategic indifference price does not equal the expectation of the derivative payoff under a suitably chosen measure. Instead, there is a nonlinear transformation of the derivative payoff through the function $\hat{f}$, which has important economic underpinning.

This follows from the fact that unlike frictionless markets, a large investor has incentive to trade the underlying to manipulate the derivative payoff and thus favorably influence the associated indifference price. This is precisely the effect captured by the function $\hat{f}$, which factors in the optimal trade off between derivative manipulation and the resulting trading cost incurred as a result of the associated price impact.

The logarithmic transformation of the expectation itself is a consequence of the assumption of exponential utility function. It suffices to illustrate that the notion of utility indifference price adds another layer of nonlinear distortion corresponding to the utility function of the investor. An immediate consequence of these distortions is that unlike the Black–Scholes model, the derivative price is no longer linear in the number of units λ.

In order to better understand the economic import of the pricing functional we have obtained, as well as its relation to the canonical model, we employ the Feynman–Kac formula to derive a pricing equation corresponding to the logarithmic pricing functional above. To this end, note that if $\mathrm{V}^i(S,t)$ denotes the time t value of the derivative for investor i, given time t underlying price S, then

$$\frac{\partial \mathrm{V}^i(S,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 \mathrm{V}^i(S,t)}{\partial S^2} + \frac{\left(\gamma^i \sigma m\right)^2}{2}\left(\frac{\partial \mathrm{V}^i(S,t)}{\partial S}\right)^2 = 0$$

In view of the equation above, we see that in the limit as $\gamma^i \downarrow 0$, the pricing equation corresponds to the canonical pricing equation. That is, in the special instance when investor i is risk–neutral, we recover the standard pricing equation. Strictly speaking, the pricing equation in the risk–neutral limit yields the Bachelier price, since the unperturbed price is driven by an arithmetic Brownian motion. Nevertheless, the analysis extends to the case when the unperturbed price is a geometric Brownian motion like the Black–Scholes model.

Additionally, given the terminal condition of the auxiliary problem (6.3), we see that though a large investor optimally manipulates underlying price with a view to influence the derivative payoff, non–manipulation is always a feasible strategy. In that case, the problem reduces to pricing a non-manipulable derivative. Further, since the underlying price volatility is constant in our model, this price corresponds with the canonical price ((Lions and Lasry, 2007, Corollary 3.1)). Thus, we have the following corollary to Proposition 6.1

Corollary 6.1. Given the strategic indifference pricing equation of investor i (5.3) for a derivative security, we have

(i) The risk–neutral strategic indifference price coincides with the canonical price.

(ii) For a manipulable derivative security, we have bid–price $\leq$ canonical price $\leq$ ask price

The form of the pricing equation resembles the classical pricing equation save for the third term on the left–hand side, which represents a correction for illiquidity arising on account of strategic competition for liquidity. We refer to this term as the *liquidity correction factor* and pay a closer attention to its implications for derivative pricing in non–Walrasian markets. The nonlinear relation between derivative holding m and the indifference price is evident from the liquidity correction factor.

Further, observe that the liquidity correction factor has a positive dependence on the volatility of the underlying asset σ. This implication is in line with empirical literature on derivative pricing, see Carr and Wu (2009) and Drechsler, Moreira and Savov (2020), which documents co–movement of liquidity premium with volatility. It seems reasonable then that strategic competition for liquidity on account of price impact will be more pronounced in volatile underlying markets and thus the liquidity correction factor associated with the price of a derivative written on such an underlying would be higher. Further, the pricing equation confirms positive dependence of liquidity correction on risk–aversion, as documented empirically in Bongaerts, De Jong and Driessen (2011) for example.

Unlike volatility and risk–aversion, which emerge as usual suspects, the liquidity correction factor also establishes a positive dependence between liquidity adjusted derivative price and the derivative delta ($\partial V/\partial S$). Moreover, it is evident from the equation that it is only the magnitude of derivative delta which affects liquidity correction and not its sign. That is, controlling for risk–aversion, volatility and holding size, the liquidity correction factor will be the same for a European option with a given strike regardless of whether the option is a call or put.

To argue the plausibility of this finding, we recall that as a rule of thumb the moneyness of a derivative is typically considered as a reasonable approximation of the delta of a derivative. Roughly, moneyness measures the worth of a derivative at the current underlying price. In line with this rule of thumb, one would expect liquidity correction to be more relevant for deep-in-the money European options as compared to at-the-money or out-of-money options. This intuition is indeed validated by empirical works such as Martin (2017), which document that deep-in-the-money European options are typically illiquid.

7. Numerical Results

In order to solve for utility indifference price of a derivative security when investors are permitted to employ general Markovian trading strategies, we adapt the numerical algorithm proposed in Achdou, Han, Lasry, Lions and Moll (2022) for solving a coupled system of nonlinear partial differential equations. Unlike their work, where the coupled system consists of a *backward* Hamilton–Jacobi–Bellman equation and a *forward* Kolomogorov–Fokker–Planck equation, in the present work we need to only consider a system of coupled backward Hamilton–Jacobi–Bellman equations.

The algorithm proposed in Achdou, Han, Lasry, Lions and Moll (2022) is a natural starting point for numerically solving the coupled system of Hamilton–Jacobi–Bellman equations characterizing the indifference price. This is because their algorithm accommodates viscosity solutions of Hamilton–Jacobi–Bellman equations by employing finite difference methods developed for viscosity solutions of nonlinear second–order partial differential equations in the seminal work Barles and Souganidis (1991). We briefly outline the key steps involved in the numerical computation of the strategic indifference price below. To this end, note that the indifference bid price of a derivative security is computed by solving the following pair of equations

$$\sup_{\vartheta\in\mathbb{R}}\left\{\left\langle D_2 F^{i,\lambda}\left(t, Z^i_t; X^{-i}\right), \beta^i\left(\vartheta^i, Z^i_t\right) x^{-i}\right\rangle + \frac{1}{2}\operatorname{tr}\left(D_2^2 F^{i,\lambda}\left(t, Z^i_t; X^{-i}\right)\chi^i\left(\vartheta^i, Z^i_t\right)\chi^i\left(\vartheta^i, Z^i_t\right)^{\mathcal{T}}\right)\right\}$$
$$+ D_1 F^{i,\lambda}\left(t, Z^i_t; X^{-i}\right) = 0$$

$$\sup_{\vartheta\in\mathbb{R}}\left\{\left\langle D_2 F^{-i,0}\left(t,Z_t^{-i};X^i\right),\beta^{-i}\left(\vartheta^{-i},Z_t^{-i}\right)x^i\right\rangle+\frac{1}{2}\mathrm{tr}\left(D_2^2F^{-i,0}\left(t,Z_t^{-i};X^i\right)\chi^{-i}\left(\vartheta^{-i},Z_t^{-i}\right)\chi^{-i}\left(\vartheta^{-i},Z_t^{-i}\right)^{\mathcal{T}}\right)\right\}$$
$$+D_1 F^{-i,0}\left(t,Z_t^{-i};X^i\right)=0$$

As a first step, we guess a trading strategy for both the investors and solve the pair of Hamilton–Jacobi–Bellman equations, which gives the best–response value function of each investor. Once we have a pair of best–response value functions and corresponding optimal *auxiliary* strategies, in the second step we reiterate the above solution procedure until we reach a Markov–Nash equilibrium.

However, there are important differences we must account for while adapting the finite difference scheme in Achdou, Han, Lasry, Lions and Moll (2022). First, unlike their work where the state is a uni-dimensional diffusion, the law of motion for our state vector is governed by a multi-dimensional Itô process. This adds significantly to the computational demands required of the numerical method. Second, the two coupled equations in their paper act as transpose of each other, which is computationally convenient. However, in the present work we must solve two coupled partial differential equations which result from optimization by heterogeneous agents.

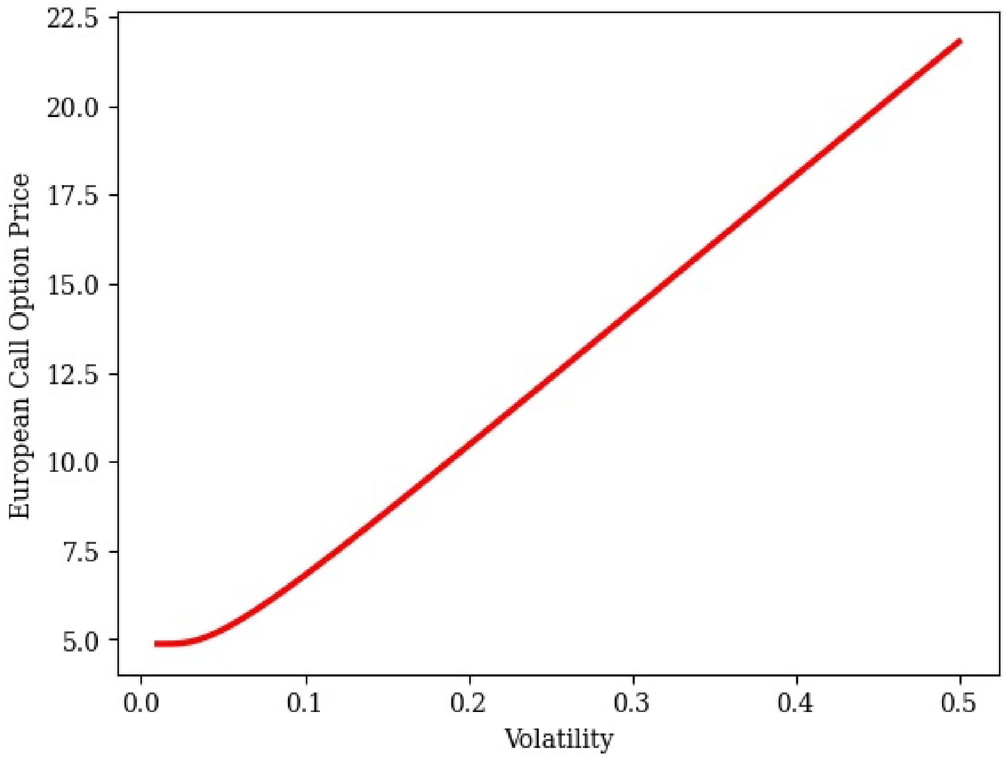

Figure 4: European Call option strategic indifference price as a function of underlying asset volatility

Figure 4 above depicts the indifference bid price of an at–the–money European call option whose current price equals 100 as a function of underlying asset volatility σ. Note that the assumption of option being at–the–money in turn implies that the strike price is also 100. The risk–aversion coefficient of the two investors equals 2 each, while their price impact parameters are taken to be $\theta^1 = 0.3$, and $\theta^2 = 0.2$ respectively. The plot confirms that even in the case when both investors are permitted the use of general Markovian trading strategies, the call price is monotonic in the underlying asset volatility. This finding remains robust to alternate parameter values; while a similar relationship holds for European puts.

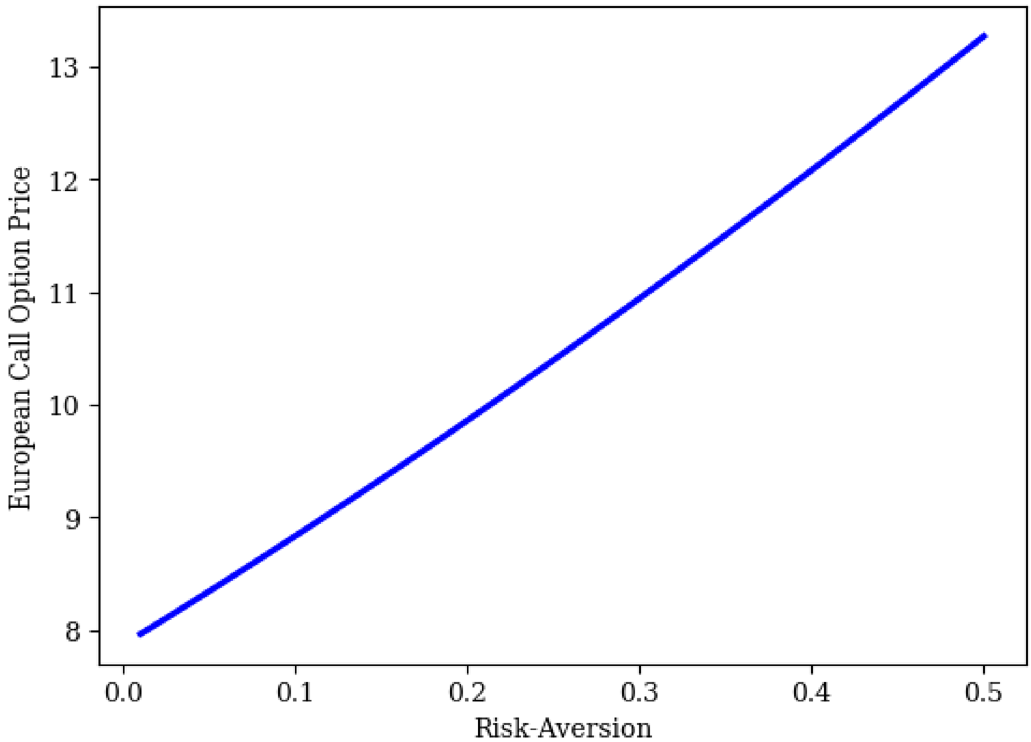

Figure 5: European Call option strategic indifference price as a function of risk–aversion coefficient

Figure 5 above also depicts the indifference bid price of an at-the-money European call option whose current price equals 100, but as a function of the risk–aversion coefficient of investor 1. As earlier, the strike price of the option will equal 100. The risk–aversion coefficient of investor 2 is taken to be 2, while the respective price impact parameters are taken to be $\theta^1 = 0.3$, and $\theta^2 = 0.2$, as earlier. From the plot, it is evident that when both investors are permitted the use of general Markovian trading strategies, we find that the call price is still monotonic in risk–aversion coefficient. Since, risk–aversion coefficient only enters the liquidity correction factor, we can infer from the figure that the liquidity correction factor grows monotonically with an investor's risk–aversion coefficient.

8. Concluding Remarks

In the present work we extend the canonical Black–Scholes option pricing model by incorporating non–Walrasian trading by large investors whose trading influences the underlying asset price on account of limited liquidity. We show limitations of arbitrage pricing in this setup by showing that the payoff space and the no–arbitrage pricing functional are convex but not necessarily linear. In a dynamic strategic framework we derive a weakly manipulation free pricing rule through certainty equivalent principle and show that the strongly manipulation free price coincides with the Black–Scholes price. We derive a liquidity adjusted Black–Scholes equation for high frequency trading paradigm and consider analytical as well as numerical computation of the price of European style options.

As a potential extension note that in principle we can generalize the setup above by considering the dual problem associated with the best–response problem of an investor which can be formulated as a minimization problem over a collection of admissible state–price densities or martingale probability measures, see Henderson and Hobson (2009, Section 2.5.5) for an illustration of the *dual* *approach* in the context of utility based pricing and hedging of derivatives written on a non–traded underlying asset. The primary advantage of the duality approach is that it allows consideration of stochastic optimal control problems which are not necessarily Markovian. However, a rigorous treatment of duality theory in characterizing the optimal behavior of utility–maximizing agents in a strategic framework remains elusive and in need of further development which is beyond the scope of the present work.

Similarly, we have chosen to sidestep the issue of potential multiplicity of Nash equilibria of the Black–Scholes–Cournot game. It seems reasonable to inquire if perhaps one can work with refinements of Nash equilibrium such as *subgame perfect* Nash equilibrium which is based on the idea of sequential rationality, or *forward induction equilibrium* which similarly assumes stringent rational behaviour on the part of players in order to exclude non–credible Nash equilibria and generalize the setup considered here. However, given the inherent challenge in extending these concepts to the case of stochastic differential games, we focus exclusively on Markov–Nash equilibrium in the present work while suggesting this line of inquiry as an interesting avenue for future research.

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A. Appendices

A.1 Motivating Example With Bounded Prices

Recall that traded price of the risky asset is assumed to be given by $\hat{S}_0(S_0, \Delta_0)$. We work under the assumption that $\hat{S}_0 = S_0 + \lambda S_0 f(\Delta_0)$, where Δ_0 is risky holding of the large investor, and the function f captures the impact of the large investor's order-flow on the traded price of the risky asset. For simplicity, we normalize the market depth parameter λ as 1 and consider the price impact function $f(\Delta_0) = \Delta_0/(1 + |\Delta_0|)$. Under these specifications the traded price of the risky asset at date 0 is bounded as depicted in the figure below

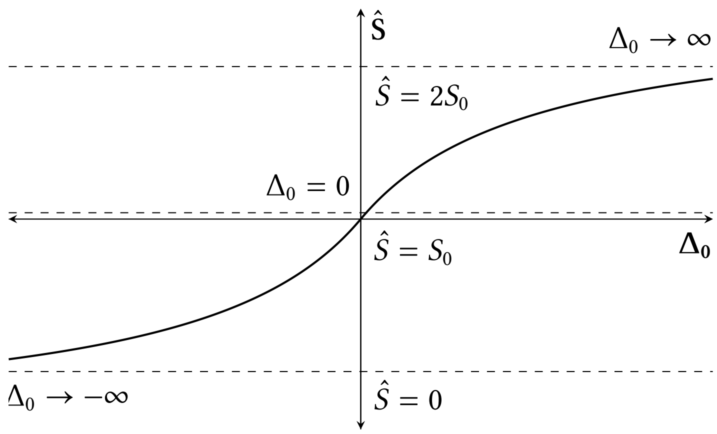

Figure 6: Traded Price For The Risky Asset In Binomial Model With Price Impact

Note: The figure plots the traded price $\hat{S}_0 = S_0 + \lambda S_0 f(\Delta_0)$, represented by the solid curve, for a representative two–period binomial model with price impact, with date 0 unperturbed price denoted by S_0, the price impact parameter $\lambda = 1$, and price impact function $f(\Delta_0) = \Delta_0/(1 + |\Delta_0|)$.

As in the case of motivating example presented in the main text, we seek to construct a replication portfolio for a contingent claim C_1 whose payoff profile across the two states is given by $C_1^{\uparrow} = 0$ and $C_1^{\downarrow} = 20$. To this end, we recall that in view of the functional form of the price impact function f, the risky asset price evolves stochastically as depicted in the figure below

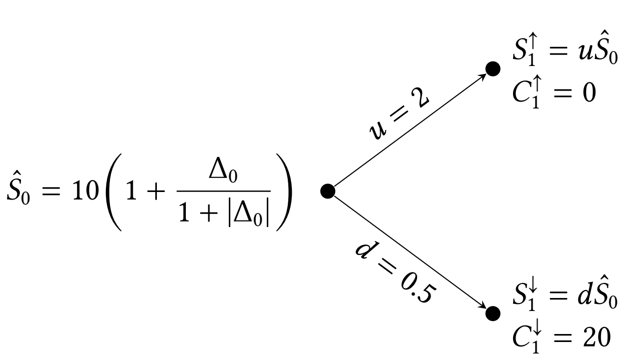

Figure 7: Two–Period Binomial Model With Price Impact

Suppose that there exists a portfolio consisting of R_0^f units of the risk–free asset and Δ_0 units of the risky asset which is able to replicate the risky payoff profile of the contingent claim C_1. Again assuming no price impact at date 1, it then follows that R_0^f and Δ_0 must solve the system of equations (A.1). We can eliminate R_0^f from the pair of equations (A.1) to get an equation in Δ_0 alone, from which it is immediate that the system has no real–valued solution, ruling out the existence of a replicating portfolio.

$$\begin{aligned} R_0^f + 20\Delta_0\left(1 + \frac{\Delta_0}{1 + |\Delta_0|}\right) &= C_1^{\uparrow} = 0 \\ R_0^f + 5\Delta_0\left(1 + \frac{\Delta_0}{1 + |\Delta_0|}\right) &= C_1^{\downarrow} = 20 \end{aligned} \tag{A.1}$$

A.2 Mathematical Appendix

We state the following technical lemma in which we recollect certain continuity properties of the controlled state process Y^i. Note that since we have assumed a constant volatility for the price of the risky asset, the lemma below serves as a corollary to (Gupta and Jacka, 2023, Lemma B.1) and we direct interested readers to consult the cited reference for a detailed proof.

Lemma A.1. Consider a deterministic initial time κ bounded above by T, and admissible trading strategies X^i and X^{-i} for investor i and $-i$ respectively. Given a positive integer $j \in \mathbb{N}$, define $\mathbb{F}$–stopping times $\tau_j^i, \tau_j^{-i}, \tau_j$ as follows

$$\tau_j^i = \inf\left\{s > \kappa \,:\, \int_\kappa^s \left|x_t^i\right| dt > j\right\},\ \tau_j^{-i} = \inf\left\{s > \kappa \,:\, \int_\kappa^s \left|x_t^{-i}\right| dt > j\right\},\ \tau_j = \tau_j^i \wedge \tau_j^{-i}$$

(i) For the stochastic process Y^i defined as the solution to multivariate stochastic differential equation (3.6), with a deterministic initial condition $Y_\kappa^i = \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}$, we can find a positive constant K, dependent on m and T, such that we have

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]} \left|Y_{s\wedge\tau_j}^i\right|^4\right] \le K\left(1 + \left|Y_\kappa^i\right|^4\right)$$

(ii) For stochastic processes Y^i and $\hat{Y}^i$ defined as solution to multivariate stochastic differential equation (3.6), with deterministic initial conditions Y_κ^i and $\hat{Y}_\kappa^i$ respectively, we can find a positive constant K, dependent on m and T, such that we have

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]} \left|Y_s^i - \hat{Y}_s^i\right|^2\right] \le K\left|Y_\kappa^i - \hat{Y}_\kappa^i\right|^2$$

(iii) For stochastic processes $Y^{i,\kappa}$ and $Y^{i,\hat{\kappa}}$ defined as solution to multivariate stochastic differential equation (3.6), for deterministic initial times time $\kappa, \hat{\kappa}$ such that $0 \le \hat{\kappa} \le \kappa \le T$ and deterministic initial condition $Y_\kappa^{i,\kappa} = Y_{\hat{\kappa}}^{i,\hat{\kappa}}$, we have

$$\lim_{\kappa\downarrow\hat{\kappa}} \mathbb{E}\left[\left|Y_T^{i,\kappa} - Y_T^{i,\hat{\kappa}}\right|^2\right] = 0$$

A.3 Proof of Lemma 5.1

Proof. (i) In order to prove the claim, we only need to show that $J^{i,\lambda} > -\infty$. To this end, consider the deterministic trading strategy $X^{i,0}$ for investor i, with $x_s^{i,0} = 0$, for $\kappa \le s \le T$. It is straightforward to check that $X^{i,0} \in \mathcal{A}^{\mathrm{m}}_{\kappa}$. Suppose, $W^{i,0}_T$ denotes the terminal cash account value of investor i associated with trading strategy $X^{i,0}$ when investor $-i$ selects admissible trading strategy X^{-i}, it follows in view of (3.3) that we have

$$\mathbb{E}\left[u^i\left(W^{i,0}_T + \lambda C_T\right)\right] = \mathbb{E}\left[-\frac{1}{\gamma^i}\exp\left(-\gamma^i W^i_{\kappa} - \gamma^i \lambda C_T\right)\right]$$

From the equation above, it is immediate that we can find a positive constant K such that we can rewrite the equality above as follows

$$\mathbb{E}\left[u^i\left(W^{i,0}_T + \lambda C_T\right)\right] = -K\mathbb{E}\left[\exp\left(-\gamma^i \lambda C_T\right)\right]$$

Further, note that in view of Assumption 5.1 it follows that we have

$$0 \le \mathbb{E}\left[\exp\left(-\gamma^i \lambda C_T\right)\right] < \infty$$

It thus follows that we have $J^{i,\lambda}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right) \ge \mathbb{E}\left[u^i\left(W^{i,0}_T + \lambda C_T\right)\right] > -\infty$

(**ii**) It follows from the definition of indirect utility functions $J^{i,\lambda,n}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right)$ and $J^{i,\lambda}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right)$ that we have

$$\limsup_{n\to\infty} J^{i,\lambda,n}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right) \le J^{i,\lambda}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right) \tag{A.2}$$

We select an ϵ−optimal hedging strategy for investor i's best−response problem, given $\epsilon > 0$, where the existence of an admissible ϵ−optimal hedging strategy follows as a consequence of (**i**) above. Observe that by definition an admissible ϵ−optimal hedging strategy $X^{i,\epsilon}$ satisfies the following

$$J^{i,\lambda}\left(\kappa, \left[S_{\kappa}, \Delta^i_{\kappa}, \Delta^{-i}_{\kappa}, W^i_{\kappa}, W^{-i}_{\kappa}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \le u^i\left(W^{i,\epsilon}_T + \lambda C\left(S^{\epsilon}_T\right)\right) \tag{A.3}$$

Note that $W^{i,\epsilon}_T$ and S^{ϵ}_T in the equation above represent respectively the cash account value of investor i and the risky asset price at terminal time T, corresponding to the choice of hedging strategies $X^{i,\epsilon}$ and X^{-i} by investor i and investor $-i$ respectively. Next, we define a sequence of admissible hedging strategies $\{X^{i,\epsilon_n}\}_{n\in\mathbb{N}}$ derived from $X^{i,\epsilon}$ as follows

$$x^{i,\epsilon_n}_s = \begin{cases} x^{i,\epsilon}_s, & \text{if } \left|x^{i,\epsilon}_s\right| \le n \\ n, & \text{if } \left|x^{i,\epsilon}_s\right| > n \end{cases}$$

It follows from the definition above that X^{i,ϵ_n} is $\mathbb{P}$−almost surely dominated pointwise by $X^{i,\epsilon}$. Moreover, X^{i,ϵ_n} also converges pointwise to $X^{i,\epsilon}$, $\mathbb{P}$−almost surely as $n \to \infty$. In view of this, we can establish the following as a straightforward consequence of dominated convergence theorem

$$\lim_{n\to\infty} \mathbb{E}\left[\int_{\kappa}^{T} \left|x^{i,\epsilon_n}_s - x^{i,\epsilon}_s\right| ds\right] = 0, \ \mathbb{P}\text{−almost surely} \tag{A.4}$$

Next, we define the controlled state process Y^{i,ϵ_n} to be the strong solution of the multivariate stochastic differential equation (3.6) associated with the choice of hedging strategies X^{i,ϵ_n} and X^{-i} by investor i and investor $-i$ respectively. Further, given $j \in \mathbb{N}$ we define $\mathbb{F}$−stopping times τ^i_j, τ^{-i}_j and τ_j as follows

$$\tau^i_j = \inf\left\{s > \kappa \ : \ \int_{\kappa}^{s} \left|x^{i,\epsilon}_t\right| dt > j\right\}, \ \tau^{-i}_j = \inf\left\{s > \kappa \ : \ \int_{\kappa}^{s} \left|x^{-i}_t\right| dt > j\right\}, \ \tau_j = \tau^i_j \wedge \tau^{-i}_j \tag{A.5}$$

Since X^{i,ϵ_n} is dominated pointwise by $X^{i,\epsilon}$ $\mathbb{P}$–almost surely, it follows that by definition we have

$$\int_{\kappa}^{s\wedge\tau_j} \left| x_t^{i,\epsilon_n} \right| dt \leq \int_{\kappa}^{s\wedge\tau_j} \left| x_t^{i,\epsilon} \right| dt \leq j \tag{A.6}$$

Given the setup above we show that Y^{i,ϵ_n} converges uniformly in expectation to $Y^{i,\epsilon}$ as $n \to \infty$, where $Y^{i,\epsilon}$ denotes the solution of multivariate stochastic differential equation (3.6) associated with the choice of hedging strategies $X^{i,\epsilon}$ and X^{-i} by investor i and investor $-i$ respectively. To this end, recall the Lipschitz continuity of $\mathbf{a}^i$ and $\mathbf{b}^i$ to ascertain that by definition of Y^{i,ϵ_n} and $Y^{i,\epsilon}$ we can find a positive constant K such that we have

$$\begin{aligned} \left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right| \leq K \int_{\kappa}^{s\wedge\tau_j} \left|x_t^{-i}\right| \left| Y_t^{i,\epsilon} - Y_t^{i,\epsilon_n} \right| dt + K \left| \int_{\kappa}^{s\wedge\tau_j} \left(\mathbf{v}^i\left(Y_t^{i,\epsilon}\right) - \mathbf{v}^i\left(Y_t^{i,\epsilon_n}\right) \right) dB_t \right| \\ + K \int_{\kappa}^{s\wedge\tau_j} \left(1 + \left|Y_t^{i,\epsilon}\right| \right) \left| x_t^{i,\epsilon} - x_t^{i,\epsilon_n} \right| dt + K \int_{\kappa}^{s\wedge\tau_j} \left|x_t^{i,\epsilon_n}\right| \left| Y_t^{i,\epsilon} - Y_t^{i,\epsilon_n} \right| dt \end{aligned} \tag{A.7}$$

Next, with notational convenience in mind, we define random variables h_s^n and l_s^n as follows

$$\begin{aligned} h_s^n &= K \left| \int_{\kappa}^{s} \left(\mathbf{v}^i\left(Y_t^{i,\epsilon}\right) - \mathbf{v}^i\left(Y_t^{i,\epsilon_n}\right) \right) dB_t \right| \\ l_s^n &= K \int_{\kappa}^{s} \left(1 + \left|Y_t^{i,\epsilon}\right| \right) \left| x_t^{i,\epsilon} - x_t^{i,\epsilon_n} \right| dt \end{aligned}$$

The notation introduced above then allows us to rewrite (A.7) succinctly as follows

$$\left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right| \leq \left(h_{s\wedge\tau_j}^n + l_{s\wedge\tau_j}^n \right) + K \int_{\kappa}^{s\wedge\tau_j} \left(\left|x_t^{i,\epsilon_n}\right| + \left|x_t^{-i}\right| \right) \left| Y_t^{i,\epsilon} - Y_t^{i,\epsilon_n} \right| dt$$

Further, invoking Gronwall's lemma we can deduce the following from the equation above, where K denotes a positive constant

$$\left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right| \leq \left(h_{s\wedge\tau_j}^n + l_{s\wedge\tau_j}^n \right) + K \int_{\kappa}^{s\wedge\tau_j} \left(\left|x_t^{i,\epsilon_n}\right| + \left|x_t^{-i}\right| \right) \left(h_t^n + l_t^n \right) dt$$

In view of the equation above and (A.6), we can again assert the existence of a positive constant K such that we have

$$\sup_{s\in[\kappa,T]} \left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right| \leq K \sup_{s\in[\kappa,T\wedge\tau_j]} \left(h_s^n + l_s^n \right)$$

Next, we recall the definition of random variables h_s^n and l_s^n, in conjunction with Cauchy–Schwarz inequality to claim the existence of a positive constant K such that the following holds

$$\begin{aligned} \mathbb{E}\left[\sup_{s\in[\kappa,T]} \left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right|^2 \right] \leq K\mathbb{E}\left[\int_{\kappa}^{T\wedge\tau_j} \left(1 + \left|Y_t^{i,\epsilon}\right| \right) \left| x_t^{i,\epsilon} - x_t^{i,\epsilon_n} \right| dt \right]^2 \\ + K\mathbb{E}\left[\sup_{s\in[\kappa,T\wedge\tau_j]} \left| \int_{\kappa}^{s} \left(\mathbf{v}^i\left(Y_t^{i,\epsilon}\right) - \mathbf{v}^i\left(Y_t^{i,\epsilon_n}\right) \right) dB_t \right| \right]^2 \end{aligned}$$

Recalling the definition of $\mathbf{v}^i$, it is immediate that the function $\mathbf{v}^i$ is Lipschitz continuous. Thus, we call upon Hölder's inequality in conjunction with Burkholder–Davis–Gundy inequality and (A.6) to find a positive constant K such that the following holds

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]} \left| Y_{s\wedge\tau_j}^{i,\epsilon} - Y_{s\wedge\tau_j}^{i,\epsilon_n} \right|^2 \right] \leq K\mathbb{E}\left[\int_{\kappa}^{T\wedge\tau_j} \left| Y_t^{i,\epsilon} - Y_t^{i,\epsilon_n} \right|^2 dt \right]$$

$$+ K\sqrt{\mathbb{E}\left[1+\sup_{s\in[\kappa,T]}\left|Y^{i,\epsilon}_{s\wedge\tau_j}\right|^4\right]}\sqrt{\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|x^{i,\epsilon}_t - x^{i,\epsilon_n}_t\right|dt\right]}$$

It is immediate from the above in view of Tonelli's theorem, Lemma A.1 and (A.6) that we have

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]}\left|Y^{i,\epsilon}_{s\wedge\tau_j} - Y^{i,\epsilon_n}_{s\wedge\tau_j}\right|^2\right] \le K\int_\kappa^T \mathbb{E}\left[\sup_{t\in[\kappa,T]}\left|Y^{i,\epsilon}_{t\wedge\tau_j} - Y^{i,\epsilon_n}_{t\wedge\tau_j}\right|^2\right]dt$$
$$+ K\sqrt{1+\left|Y^{i,\epsilon}_\kappa\right|^4}\sqrt{\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|x^{i,\epsilon}_t - x^{i,\epsilon_n}_t\right|dt\right]}$$

The desired convergence of Y^{i,ϵ_n} to $Y^{i,\epsilon}$ on the interval $[\kappa,\tau_j\wedge T]$ follows from the equation above as $n\to\infty$, in view of Gronwall's lemma and (A.4). Since, the choice of j above was arbitrary, and the trading strategy X^{-i} as well as X^{i,ϵ_n} are admissible in the sense of Definition 3.2 for $n\in\mathbb{N}$, the convergence holds on the interval $[\kappa,T]$ in the limit as $j\to\infty$, $\mathbb{P}$–almost surely. Next, we employ the convergence result to prove the claim in the statement of the lemma. That is, we need to show

$$\lim_{n\to\infty}\mathbb{E}\left[u^i\left(W^{i,\epsilon_n}_T + \lambda C\left(S^{\epsilon_n}_T\right)\right)\right] = \mathbb{E}\left[u^i\left(W^{i,\epsilon}_T + \lambda C\left(S^{\epsilon}_T\right)\right)\right]$$

To this end, we recall the convergence result proved above which in conjunction with continuity of the utility function u^i and Assumption 5.1 implies that we have $\mathbb{P}$–almost surely

$$u^i\left(W^{i,\epsilon_n}_T + \lambda C\left(S^{\epsilon_n}_T\right)\right) \to u^i\left(W^{i,\epsilon}_T + \lambda C\left(S^{\epsilon}_T\right)\right) \text{ as } n\to\infty$$

In order to show that the pointwise convergence of utilities implies convergence of expected utilities we show that the sequence of utilities associated with bounded hedging strategies is uniformly integrable. To this end, we note that in view of the exponential form of the utility function and an iterated application of Cauchy–Schwarz inequality we can ascertain the existence of a positive constant K such that we have

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} u^i\left(W^{i,\epsilon_n}_T + \lambda C\left(S^{\epsilon_n}_T\right)\right)\right] \le K\left[\sup_{n\in\mathbb{N}}\exp\left(K\int_\kappa^T x^{i,\epsilon_n}_s\,ds\right)\right]\times\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(K\int_\kappa^T \Delta^{i,\epsilon_n}_s x^{i,\epsilon_n}_s\,ds\right)\right]$$
$$\times\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(K\int_\kappa^T \Delta^{-i}_s x^{i,\epsilon_n}_s\,ds\right)\right]\times\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(K\int_s^T x^{i,\epsilon_n}_s B_s\,ds\right)\right]$$
$$\times\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(KC\left(S^{\epsilon_n}_T\right)\right)\right]$$

Recall that $|x^{i,\epsilon_n}_s| \le |x^{i,\epsilon}_s|$ for all $n\in\mathbb{N}$ and $s\in[\kappa,T]$ by definition, which in conjunction with Definition 3.2 implies that the first term on the right–hand side above is finite. Similarly, we show that the remaining terms on the right–hand side of the equation above are finite as well. To this end, observe that by Hölder's inequality, and the definition of $\Delta^{i,\epsilon}$ we can bound the second term on the right–hand side of the equation above as follows for some positive constant $\hat{K}$

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(K\int_\kappa^T \Delta^{i,\epsilon_n}_s x^{i,\epsilon_n}_s\,ds\right)\right] \le \hat{K}\mathbb{E}\left[\sup_{n\in\mathbb{N}}\exp\left(\hat{K}\left(\sup_{s\in[\kappa,T]}\left|\Delta^{i,\epsilon_n}_s\right|\right)\int_\kappa^T\left|x^{i,\epsilon_n}_s\right|ds\right)\right]$$
$$\le \hat{K}\mathbb{E}\left[\exp\left(\hat{K}\int_\kappa^T\left|x^{i,\epsilon}_s\right|ds\int_\kappa^T\left|x^{i,\epsilon}_s\right|ds\right)\right]$$

Given that the hedging strategy $X^{i,\epsilon}$ is admissible in the sense of Definition 3.2 it follows immediately that the right–hand side of the inequality above is finite. We again employ Hölder's inequality in conjunction with the definition of Δ^{-i} to obtain the following upper bound for some positive constant $\hat{K}$

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \Delta_s^{-i}\, x_s^{i,\epsilon_n}\, ds\right)\right] \leq \hat{K}\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(\hat{K}\left(\sup_{s\in[\kappa,T]}\left|\Delta_s^{-i}\right|\right)\int_\kappa^T \left|x_s^{i,\epsilon_n}\right| ds\right)\right]$$
$$\leq \hat{K}\mathbb{E}\left[\exp\left(\hat{K}\int_\kappa^T \left|x_s^{-i}\right| ds \int_\kappa^T \left|x_s^{i,\epsilon}\right| ds\right)\right]$$

Recalling the fact that the hedging strategies $X^{i,\epsilon}$ and X^{-i} are both admissible we ascertain that the term on the right–hand side of the inequality above is finite. Next, we use Hölder's inequality, along with (Mörters and Peres, 2010, Theorem 2.21) which ascertains that the running maximum of standard Brownian motion has a half–normal distribution, with finite expectation on account of the fact that $T < \infty$, to establish the existence of a positive constant K such that

$$\mathbb{E}\left[\exp\left(K\int_\kappa^T x_s^{i,\epsilon}\, B_s\, ds\right)\right] \leq \mathbb{E}\left[\exp\left(K\left(\sup_{s\in[\kappa,T]} B_s\right)\int_\kappa^T \left|x_s^{i,\epsilon}\right| ds\right)\right]$$
$$\leq K\mathbb{E}\left[\exp\left(K\int_\kappa^T \left|x_s^{i,\epsilon}\right| ds\right)\right]$$

As earlier, since the hedging strategy $X^{i,\epsilon}$ is admissible by hypothesis, it follows that the right–hand side of the inequality above is finite. Therefore, as a consequence of the arguments above we arrive at the following convergence result

$$\lim_{n\to\infty} \mathbb{E}\left[u^i\left(W^{i,\epsilon_n} + \lambda C(S^{\epsilon_n})\right)\right] = \mathbb{E}\left[u^i\left(W_T^{i,\epsilon} + \lambda C(S^{\epsilon})\right)\right]$$

Moreover, by combining the equation above with (A.3), we note that the following holds true

$$\lim_{n\to\infty} \mathbb{E}\left[u^i\left(W^{i,\epsilon_n} + \lambda C(S^{\epsilon_n})\right)\right] = \mathbb{E}\left[u^i\left(W_T^{i,\epsilon} + \lambda C(S^{\epsilon})\right)\right] \geq J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon$$

Further, observe that given $q, n \in \mathbb{N}$, with $n \leq q$, we have $\mathcal{A}_\kappa^{\mathrm{m},n} \subseteq \mathcal{A}_\kappa^{\mathrm{m},q}$ by definition, which in turn implies that $J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) \leq J^{i,\lambda,q}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right)$ and thus we have

$$\mathbb{E}\left[u^i\left(W^{i,\epsilon_n} + \lambda C(S^{\epsilon_n})\right)\right] \leq \sup_{q\geq n} J^{i,\lambda,q}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right)$$

Taking the limit as $n \to \infty$ in the equation above, we arrive at the following by way of (A.2)

$$J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \leq \limsup_{n\to\infty} \mathbb{E}\left[u^i\left(W^{i,\epsilon_n} + \lambda C(S^{\epsilon_n})\right)\right]$$
$$\leq \limsup_{n\to\infty} J^{i,\lambda,n}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right)$$
$$\leq J^{i,\lambda}\left(\kappa, \left[S_\kappa, \Delta_\kappa^i, \Delta_\kappa^{-i}, W_\kappa^i, W_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right)$$

The claim then follows immediately from the above given that the choice of ϵ was arbitrary.

(**iii**) In order to prove the claim, we consider a sequence of deterministic initial state vectors $[S^n_\kappa, \Delta^{i,n}_\kappa, \Delta^{-i,n}_\kappa, W^{i,n}_\kappa, W^{-i,n}_\kappa]^{\mathrm{T}}$ such that $\lim_{n\to\infty}\left[S^n_\kappa, \Delta^{i,n}_\kappa, \Delta^{-i,n}_\kappa, W^{i,n}_\kappa, W^{-i,n}_\kappa\right]^{\mathrm{T}} = \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}$. Then, in view of the definition of indirect utility function $J^{i,\lambda,n}$ it follows that we have

$$\begin{aligned}
&\limsup_{n\to\infty}\left|J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - J^{i,\lambda,l}\Big(\kappa, \left[S^n_\kappa, \Delta^{i,n}_\kappa, \Delta^{-i,n}_\kappa, W^{i,n}_\kappa, W^{-i,n}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\right| \\
&\quad\le \limsup_{n\to\infty}\sup_{X^i\in\mathcal{A}^{\mathrm{m},l}_\kappa}\mathbb{E}\left[\left|u^i\big(W^i_T+\lambda C(S_T)\big) - u^i\big(W^{i,n}_T+\lambda C\big(S^n_T\big)\big)\right|\right]
\end{aligned}$$

For n large enough, we can therefore rewrite the inequality above as follows

$$\begin{aligned}
&\limsup_{n\to\infty}\left|J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - J^{i,\lambda,l}\Big(\kappa, \left[S^n_\kappa, \Delta^{i,n}_\kappa, \Delta^{-i,n}_\kappa, W^{i,n}_\kappa, W^{-i,n}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\right| \\
&\quad\le \sup_{X^i\in\mathcal{A}^{\mathrm{m},l}_\kappa}\limsup_{n\to\infty}\mathbb{E}\left[\left|u^i\big(W^i_T+\lambda C(S_T)\big) - u^i\big(W^{i,n}_T+\lambda C\big(S^n_T\big)\big)\right|\right]
\end{aligned}$$

Further, we use strict concavity of the utility function u^i along with Cauchy–Schwarz inequality and Assumption 5.1 in order to ascertain

$$\begin{aligned}
&\limsup_{n\to\infty}\mathbb{E}\left[\left|u^i\big(W^i_T+\lambda C(S_T)\big) - u^i\big(W^{i,n}_T+\lambda C\big(S^n_T\big)\big)\right|\right] \\
&\quad\le K\limsup_{n\to\infty}\mathbb{E}\left[\left|\nabla\cdot u^i\big(W^i_T+\lambda C(S_T)\big)\right|\left|W^i_T+\lambda C(S_T) - W^{i,n}_T - \lambda C\big(S^n_T\big)\right|\right] \\
&\quad\le K\limsup_{n\to\infty}\mathbb{E}\left[\left|\nabla\cdot u^i\big(W^i_T+\lambda C(S_T)\big)\right|^2\right]\mathbb{E}\left[\left|Y^i_T - Y^{i,n}_T\right|^2\right]
\end{aligned}$$

We can use a similar argument as in (**ii**) above to show that given an admissible hedging strategy X^i we have

$$\mathbb{E}\left[\left|\nabla\cdot u^i\big(W^i_T+\lambda C(S_T)\big)\right|^2\right] < \infty$$

Also, recall that as a straightforward consequence of Lemma A.1(ii), it follows that

$$\mathbb{E}\left[\left|Y^i_T - Y^{i,n}_T\right|^2\right] \le \left|\left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}} - \left[S^n_\kappa, \Delta^{i,n}_\kappa, \Delta^{-i,n}_\kappa, W^{i,n}_\kappa, W^{-i,n}_\kappa\right]^{\mathrm{T}}\right|^2$$

The claim in the statement of the lemma is then immediate from the above.

(**iv**) In order to prove the claim, we note from (**ii**) above that for every $\epsilon > 0$ we can find $n \in \mathbb{N}$ such that for $l > n$ we have

$$J^{i,\lambda}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) < \frac{\epsilon}{2}$$

Also, in view of (**iii**) above, given a deterministic state vector $[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa]^{\mathrm{T}}$ we can find $\delta > 0$ such that for $\left[\tilde S_\kappa, \tilde\Delta^{i}_\kappa, \tilde\Delta^{-i}_\kappa, \tilde W^{i}_\kappa, \tilde W^{-i}_\kappa\right]^{\mathrm{T}} \in \mathcal{B}_\delta\big(\left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}\big)$ by lower semicontinuity of $J^{i,\lambda,l}$ we have

$$J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - \frac{\epsilon}{2} < J^{i,\lambda,l}\Big(\kappa, \left[\tilde S_\kappa, \tilde\Delta^{i}_\kappa, \tilde\Delta^{-i}_\kappa, \tilde W^{i}_\kappa, \tilde W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)$$

Since, the indirect utility funtion $J^{i,\lambda,n}$ is bounded above by $J^{i,\lambda}$ the arguments above lead us to

$$\begin{aligned}
J^{i,\lambda}\Big(\kappa, \left[\tilde S_\kappa, \tilde\Delta^{i}_\kappa, \tilde\Delta^{-i}_\kappa, \tilde W^{i}_\kappa, \tilde W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) &\ge J^{i,\lambda,l}\Big(\kappa, \left[\tilde S_\kappa, \tilde\Delta^{i}_\kappa, \tilde\Delta^{-i}_\kappa, \tilde W^{i}_\kappa, \tilde W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) \\
&\ge J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - \frac{\epsilon}{2} \\
&\ge J^{i,\lambda}\Big(\kappa, \left[S_\kappa, \Delta^{i}_\kappa, \Delta^{-i}_\kappa, W^{i}_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big) - \epsilon
\end{aligned}$$

The claim then follows from the above since the choice of ϵ was arbitrary.

(**v**) We consider an increasing sequence of deterministic times $\{\kappa_n\}_{n\in\mathbb{N}} \subseteq [\kappa, T)$, such that we have $\kappa_n \uparrow T$ as $n \to \infty$. Given the definition of $J^{i,\lambda,l}$, it then follows that we have

$$\begin{aligned}&\limsup_{n\to\infty} \left| J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\big|_{\kappa=T} - J^{i,\lambda,l}\Big(\kappa_n, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\right| \\ &\quad \le \limsup_{n\to\infty} \sup_{X^i \in \mathcal{A}^{\mathrm{m},l}_\kappa} \mathbb{E}\left[\left| u^i\big(W^i_T + \lambda C(S_T)\big) - u^i\big(W^{i,n}_T + \lambda C\big(S^n_T\big)\big)\right|\right]\end{aligned}$$

Thus, given the above it follows that for n large enough we have

$$\begin{aligned}&\limsup_{n\to\infty} \left| J^{i,\lambda,l}\Big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\big|_{\kappa=T} - J^{i,\lambda,l}\Big(\kappa_n, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\Big)\right| \\ &\quad \le \sup_{X^i \in \mathcal{A}^{\mathrm{m},l}_\kappa} \limsup_{n\to\infty} \mathbb{E}\left[\left| u^i\big(W^i_T + \lambda C(S_T)\big) - u^i\big(W^{i,n}_T + \lambda C\big(S^n_T\big)\big)\right|\right]\end{aligned}$$

Further, we recall that the utility function u^i is strictly concave in conjunction with Assumption 5.1 and Cauchy–Schwarz inequality so as to obtain

$$\begin{aligned}&\limsup_{n\to\infty} \mathbb{E}\left[\left| u^i\big(W^i_T + \lambda C(S_T)\big) - u^i\big(W^{i,n}_T + \lambda C\big(S^n_T\big)\big)\right|\right] \\ &\quad \le K \limsup_{n\to\infty} \mathbb{E}\left[\left|\nabla \cdot u^i\big(W^i_T + \lambda C(S_T)\big)\right|^2\right] \mathbb{E}\left[\left|Y^i_T - Y^{i,n}_T\right|^2\right]\end{aligned}$$

Next, we can proceed as in (**ii**) above to establish that

$$\mathbb{E}\left[\left|\nabla \cdot u^i\big(W^i_T + \lambda C(S_T)\big)\right|^2\right] < \infty$$

Further, in view of Lemma A.1(iii) it also follows that we have

$$\limsup_{n\to\infty} \mathbb{E}\left[\left|W^i_T - W^{i,n}_T\right|^2\right] = 0$$

Thus, we conclude that the indirect utility function $J^{i,\lambda,l}\big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\big)\big|_{\kappa=T}$ is continuous.

The lower semicontinuity of $J^{i,\lambda}\big(\kappa, \left[S_\kappa, \Delta^i_\kappa, \Delta^{-i}_\kappa, W^i_\kappa, W^{-i}_\kappa\right]^{\mathrm{T}}; X^{-i}\big)\big|_{\kappa=T}$ follows from an argument identical to (**iv**) above. □

A.4 Proof of Lemma 5.5

Proof. Given an admissible strategy X^{-i} for investor $-i$, we define $\mathbb{F}$–stopping time τ_j where $j \in \mathbb{N}$ denotes a positive integer as

$$\tau_j = \inf\left\{ s > \kappa \,:\, \int_\kappa^s \left|x^{-i}_t\right| dt > j \right\} \tag{A.8}$$

In view of the definition of controlled auxiliary state processes $Z^{i,n}$ and Z^i, we then have

$$\left|Z^{i,n}_{s\wedge\tau_j} - Z^i_{s\wedge\tau_j}\right| \le \int_\kappa^{s\wedge\tau_j} \left|x^{-i}_t\right| \left|\boldsymbol{\beta}^i\big(\vartheta^{i,n}_t, Z^{i,n}_t\big) - \boldsymbol{\beta}^i\big(\vartheta^i_t, Z^i_t\big)\right| dt + \left|\int_\kappa^{s\wedge\tau_j} \boldsymbol{\chi}^i\big(\vartheta^{i,n}_t, Z^{i,n}_t\big) - \boldsymbol{\chi}^i\big(\vartheta^i_t, Z^i_t\big)\, dB_t\right|$$

Further, recall that by definition we have

$$\begin{aligned}\boldsymbol{\beta}^i\left(\vartheta_t^i, Z_t^i\right) &= \left[-\theta^{-i}, \, 0, \, -1, \, -\theta^{-i}\vartheta_t^i, \, \pi_t^i + \theta^i\vartheta_t^i\right]^{\mathrm{T}} \\ \boldsymbol{\chi}^i\left(\vartheta_t^i, Z_t^i\right) &= \sigma\left[1, \, 0, \, 0, \, \vartheta_t^i, \, 0\right]^{\mathrm{T}}\end{aligned}$$

In view of the above, we can assert the existence of a positive constant K such that we have

$$\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right| \leq K\int_\kappa^{s\wedge\tau_j}\left|x_t^{-i}\right|\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt + K\int_\kappa^{s\wedge\tau_j}\left|x_t^{-i}\right|\left|Z_t^{i,n} - Z_t^i\right|dt + K\left|\int_\kappa^{s\wedge\tau_j}\left(\vartheta_t^{i,n} - \vartheta_t^i\right)dB_t\right|$$

Further, for notational convenience, we define random variables h_s^n and l_s^n as follows

$$\begin{aligned}h_s^n &= K\left|\int_\kappa^{s}\left(\vartheta_t^{i,n} - \vartheta_t^i\right)dB_t\right| \\ l_s^n &= K\int_\kappa^{s}\left|x_t^{-i}\right|\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt\end{aligned}$$

With the help of the notation introduced above, we note that there exists a positive constant K such that the following holds

$$\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right| \leq \left(h_{s\wedge\tau_j}^n + l_{s\wedge\tau_j}^n\right) + K\int_\kappa^{s\wedge\tau_j}\left|x_t^{-i}\right|\left|Z_t^{i,n} - Z_t^i\right|dt$$

Further, invoking Gronwall's lemma and recalling (A.8) we can deduce the following from the equation above, where K denotes a positive constant

$$\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right| \leq \left(h_{s\wedge\tau_j}^n + l_{s\wedge\tau_j}^n\right) + K\int_\kappa^{s\wedge\tau_j}\left|x_t^{-i}\right|\left(h_t^n - l_t^n\right)dt$$

Recalling equation (A.8) above, we can again assert the existence of a positive constant K such that we have

$$\sup_{s\in[\kappa,T]}\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right| \leq \sup_{s\in[\kappa,T\wedge\tau_j]}\left(h_s^n + l_s^n\right)$$

Given the above, it then follows in view of the definition of random variables h_s^n and l_s^n that there exists a positive constant K such that the following holds

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]}\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right|^2\right] \leq K\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|x_t^{-i}\right|\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt\right]^2 + K\mathbb{E}\left[\sup_{s\in[\kappa,T\wedge\tau_j]}\left|\int_\kappa^{s}\left(\vartheta_t^{i,n} - \vartheta_t^i\right)dB_t\right|\right]^2$$

Additionally, in view of Burkholder–Davis–Gundy inequality it is immediate from the above that we have

$$\mathbb{E}\left[\sup_{s\in[\kappa,T]}\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right|^2\right] \leq K\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|x_t^{-i}\right|\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt\right]^2 + K\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|\vartheta_t^{i,n} - \vartheta_t^i\right|^2dt\right]$$

Note that since $\mathcal{V}^i$ and $\mathcal{V}^{i,n}$ are admissible in the sense of Definition 4.1, both $\left|\vartheta_t^i\right|$ as well as $\left|\vartheta_t^{i,n}\right|$ are bounded above. It then follows in view of (A.8) that we have

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[\sup_{s\in[\kappa,T]}\left|Z_{s\wedge\tau_j}^{i,n} - Z_{s\wedge\tau_j}^{i}\right|^2\right] \leq \; & K\lim_{n\to\infty}\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|x_t^{-i}\right|\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt\right] \\ & + K\lim_{n\to\infty}\mathbb{E}\left[\int_\kappa^{T\wedge\tau_j}\left|\vartheta_t^{i,n} - \vartheta_t^i\right|dt\right]\end{aligned} \tag{A.9}$$

Further, as established earlier, both $|\vartheta_t^i|$ and $|\vartheta_t^{i,n}|$ are bounded above. Thus, it follows in view of (A.8) in conjunction with Tonelli's theorem and dominated convergence theorem that we have

$$\lim_{n\to\infty} \mathbb{E}\left[\int_\kappa^{T\wedge\tau_j} |x_t^{-i}|\,|\vartheta_t^{i,n} - \vartheta_t^i|\,dt\right] = \mathbb{E}\left[\int_\kappa^{T\wedge\tau_j} \lim_{n\to\infty} |x_t^{-i}|\,|\vartheta_t^{i,n} - \vartheta_t^i|\,dt\right]$$

Given that the strategy X^{-i} is admissible in the sense of Definition 3.2, it follows that $|x_t^{-i}| < \infty$, $\mathbb{P}\otimes\lambda_{[\kappa,T]}$ almost everywhere, where $\lambda_{[s,T]}$ denotes the restriction of the Lebesgue measure defined over the real line $\mathbb{R}$ on the interval $[\kappa, T]$. Also, since $\boldsymbol{\rho}\left(\mathcal{V}^{i,n}, \mathcal{V}^i\right) \to 0$ as $n\to\infty$, we can find a subsequence $\{n_{\mathrm{p}}\}_{\mathrm{p}\in\mathbb{N}}$ such that $\left|\vartheta_t^{i,n_{\mathrm{p}}} - \vartheta_t^i\right| \to 0$, $\mathbb{P}\otimes\lambda_{[\kappa,T]}$ almost everywhere as $\mathrm{p}\to\infty$. This fact in conjunction with the equation above then implies that

$$\lim_{\mathrm{p}\to\infty} \mathbb{E}\left[\int_\kappa^{T\wedge\tau_j} |x_t^{-i}|\left|\vartheta_t^{i,n_{\mathrm{p}}} - \vartheta_t^i\right|dt\right] = \mathbb{E}\left[\int_\kappa^{T\wedge\tau_j} \lim_{\mathrm{p}\to\infty} |x_t^{-i}|\left|\vartheta_t^{i,n_{\mathrm{p}}} - \vartheta_t^i\right|dt\right] = 0$$

In view of the equation above, it is immediate that the first term on the right–hand side of (A.9) converges to zero in the limit as $\mathrm{p}\to\infty$. Also, note that by hypothesis the second term on the right–hand side of (A.9) again converges to zero in the limit as $\mathrm{p}\to\infty$.

Thus, the desired convergence holds on the interval $[\kappa, \tau_j]$ in view of preceding arguments. Further, as the strategy X^{-i} is admissible in the sense of Definition 3.2, we can ascertain that the convergence holds on the interval $[\kappa, T]$ $\mathbb{P}$–almost surely in the limit as $j\to\infty$, thus establishing the claim in the statement of the lemma. □

A.5 Proof of Lemma 5.6

Proof. Given a sequence $\{T_n\}$ of nested partitions of the time interval $[\kappa, T]$, it is straightforward to check that given $n, q\in\mathbb{N}$ such that $q\le n$, we have $\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_q) \subseteq \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)$. In view of this, it is immediate that the following holds

$$\begin{aligned}&\sup_{\mathcal{V}^{i,n}\in\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)} \mathbb{E}\left[\sup_{\varphi\in\mathfrak{R}} u^i\left(f^i\left(-\varphi, \psi_T^{i,n}\right) + \lambda C\circ f^i\left(-\varphi, \pi_T^{i,n}\right)\right)\right]\\ &\quad\ge \sup_{\mathcal{V}^{i,q}\in\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_q)} \mathbb{E}\left[\sup_{\varphi\in\mathfrak{R}} u^i\left(f^i\left(-\varphi, \psi_T^{i,q}\right) + \lambda C\circ f^i\left(-\varphi, \pi_T^{i,q}\right)\right)\right]\end{aligned}$$

Moreover, we also have $\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n) \subseteq \mathcal{A}_\kappa^{\mathrm{m},a}$ by definition, which together with the definition of $F^{i,\lambda}$ then leads us to

$$\begin{aligned}&F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right)\\ &\quad\ge \limsup_{n\to\infty} \sup_{\mathcal{V}^{i,n}\in\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)} \mathbb{E}\left[\sup_{\varphi\in\mathfrak{R}} u^i\left(f^i\left(-\varphi, \psi_T^{i,n}\right) + \lambda C\circ f^i\left(-\varphi, \pi_T^{i,n}\right)\right)\right]\end{aligned} \tag{A.10}$$

In order to prove the claim in the statement of the theorem, it remains to show that the converse is true. To this end, we shall consider the manipulable and non–manipulable cases separately.

Non–Manipulable Case – Observe that in the non–manipulable case, we have $\vartheta_T^i = \Delta_T^i = 0$ by hypothesis, which in turn implies that

$$\sup_{\varphi\in\mathfrak{R}} u^i\left(f^i\left(-\varphi, \psi_T^i\right) + \lambda C\circ f^i\left(-\varphi, \pi_T^i\right)\right) = u^i\left(\psi_T^i + \lambda C\left(\pi_T^i\right)\right)$$

Next, we show that the indirect utility function $F^{i,\lambda}$ is non–degenerate in the manipulable case, that is, we need to show that $F^{i,\lambda} > -\infty$. To this end, we consider the zero strategy $\mathcal{V}^{i,0}$ and note that $\psi_T^{i,0} = \psi_\kappa^{i,0}$. Moreover, it is immediate from the definition of auxiliary state variables that π_T^i is not affected by the strategy of investor i, which gives us

$$F^{i,\lambda}\Big(\kappa,\, \big[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big) \geq \mathbb{E}\left[u^i\big(\psi_\kappa^i + \lambda C\big(\pi_T^i\big)\big)\right]$$

In view of the exponential form of the utility function and Cauchy–Schwarz inequality, we observe from the equation above that in order to establish the non–degeneracy of $F^{i,\lambda}$ it suffices to show that given some positive constant K we have

$$K\mathbb{E}\left[\exp\left(KC\big(\pi_T^i\big)\right)\right] < \infty$$

It is straightforward to check that the above inequality follows as a direct consequence of Assumption 5.1. Next, given $\epsilon > 0$, we consider an ϵ–optimal admissible strategy $\mathcal{V}^{i,\epsilon}$ for the auxiliary hedging problem. The existence of an ϵ–optimal admissible strategy for the auxiliary hedging problem follows from the definition of $F^{i,\lambda}$. Observe that the ϵ–optimal admissible strategy $\mathcal{V}^{i,\epsilon}$ satisfies the following inequality by definition

$$F^{i,\lambda}\Big(\kappa,\, \big[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\big]^{\mathrm{T}}; X^{-i}\Big) - \epsilon \leq \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon} + \lambda C\big(\pi_T^{i,\epsilon}\big)\big)\right] \tag{A.11}$$

Note that $\psi_T^{i,\epsilon}$ and $\pi_T^{i,\epsilon}$ on the right–hand side of the equation above denote the terminal value of the corresponding auxiliary state variables when investor i and investor $-i$ employ strategies $\mathcal{V}^{i,\epsilon}$ and X^{-i} respectively. In view of Lemma 5.4, we can find a sequence of admissible strategies $\big\{\mathcal{V}^{i,\epsilon_n} \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)\big\}$ such that $\boldsymbol{\rho}(\mathcal{V}^{i,\epsilon_n}, \mathcal{V}^{i,\epsilon}) \to 0$, as $n \to \infty$. From Lemma 5.5, it follows that $Z_T^{i,\epsilon_n} \to Z_T^{i,\epsilon}$, $\mathbb{P}$-almost surely as $n \to \infty$ (at least along a subsequence), where Z^{i,ϵ_n} denotes the controlled auxiliary state process when investor i and investor $-i$ employ strategies $\mathcal{V}^{i,\epsilon_n}$ and X^{-i} respectively. Given the preceding arguments, we note that in order to prove the desired claim it suffices to show that

$$\lim_{n\to\infty} \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big)\right] = \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon} + \lambda C\big(\pi_T^{i,\epsilon}\big)\big)\right]$$

In light of arguments presented above, we conclude that the right–hand of the equation above is finite. Further, we note from the definition of the auxiliary state variables that π_T^i does not depend on the strategy of investor i, which then implies that we have $\pi_T^{i,\epsilon} = \pi_T^{i,\epsilon_n} = \pi_T^i$ for all $n \in \mathbb{N}$. Thus, as an immediate consequence of the continuity of the utility function u^i and the flow f^i it follows that we have $\mathbb{P}$–almost surely

$$u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big) \longrightarrow u^i\big(\psi_T^{i,\epsilon} + \lambda C\big(\pi_T^i\big)\big) \text{ as } n \to \infty$$

Given the above, it follows that in order to show that the sequence of expected utilities associated with piecewise constant strategies converges to expected utility from the ϵ–optimal control as $n \to \infty$, it suffices to show that the sequence of expected utilities associated with piecewise constant strategies is uniformly integrable, that is, we intend to show

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big)\right] < \infty$$

To this end, recall that the auxiliary state variable π_T^i does not depend on the strategy of investor i and note that in view of the exponential form of the utility function and iterated application of Cauchy–Schwarz inequality we can find a positive constant K such that we have

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big)\right] \leq K\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \vartheta_s^{i,\epsilon_n}\, x_s^{-i} ds\right)\right] \times \mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \vartheta_s^{i,\epsilon_n} dB_s\right)\right] \times \mathbb{E}\left[\exp\left(KC\big(\pi_T^i\big)\right)\right]$$

Given Assumption 5.1, it is straightforward to check that the third expectation on the right–hand side above is finite. It remains to show that the first two expectations on the right–hand side of the equation above are finite. Observe that by Hölder's inequality, we obtain the following upper bound for the first term on the right–hand side of the equation above for some positive constant $\hat{K}$

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \vartheta_s^{i,\epsilon}\, x_s^{-i}\, ds\right)\right] \leq \mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(\hat{K}\left(\sup_{s\in[\kappa,T]} \left|\vartheta_s^{i,\epsilon}\right|\right) \int_\kappa^T \left|x_s^{-i}\right| ds\right)\right]$$

It is straightforward to check that in view of Definition 3.2 and Definition 4.1 the right–hand side of the inequality above is finite. In a similar vein, we appeal to (Protter, 2004, Theorem 39, Page 138) in conjunction with Definition 4.1 in order to ascertain the existence of a positive constant $\hat{K}$ such that we have

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \vartheta_s^{i,\epsilon} dB_s\right)\right] \leq \mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(\hat{K}\int_\kappa^T \big(\vartheta_s^{i,\epsilon}\big)^2 ds\right)\right] < \infty$$

Thus, in view of the preceding arguments and (A.11), we then arrive at the following

$$F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \leq \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon} + \lambda C\big(\pi_T^{i,\epsilon}\big)\big)\right] = \limsup_{n\to\infty} \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big)\right]$$

Moreover, as an immediate consequence of the inequality above we ascertain that

$$F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \leq \limsup_{n\to\infty} \sup_{\mathcal{V}^{i,n}\in\mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)} \mathbb{E}\left[u^i\big(\psi_T^{i,\epsilon_n} + \lambda C\big(\pi_T^{i,\epsilon_n}\big)\big)\right]$$

The desired claim then follows by combining the inequality above with (A.10) given that the choice of ϵ was arbitrary.

Manipulable Case – In the manipulable case, we note that in view of the strict monotonicity of the utility function u^i we can write

$$\begin{aligned}\sup_{\varphi\in\Re} u^i\big(f^i\big(-\varphi,\psi_T^i\big) + \lambda C\circ f^i\big(-\varphi,\pi_T^i\big)\big) &= u^i\big(\sup_{\varphi\in\Re}\big(f^i\big(-\varphi,\psi_T^i\big) + \lambda C\circ f^i\big(-\varphi,\pi_T^i\big)\big)\big) \\ &= u^i\big(\sup_{\varphi\in\Re}\big(\psi_T^i - \pi_T^i\,\varphi - (\theta^i/2)\,\varphi^2 + \lambda C\big(\pi_T^i + \theta^i\varphi\big)\big)\big)\end{aligned}$$

In view of Assumption 5.2 it is straightforward to check that the supremum on the right–hand side of the equation above can be characterized by the first–order condition. In view of this, it follows that the associated maximizer φ^* can be specified implicitly as the solution to the following equation

$$\mathcal{G}\big(\pi_T^i,\varphi\big) = -\pi_T^i - \theta^i\varphi + \lambda\theta^i D_1 C\big(\pi_T^i + \theta^i\varphi\big) = 0 \tag{A.12}$$

Observe that given Assumption 5.2 it is straightforward to check that $D_2\mathcal{G}\big(\pi_T^i,\varphi^*\big) \neq 0$. Thus, we invoke Dini's implicit function theorem (Dontchev and Rockafellar, 2009, Chapter 1, Page 5) to ascertain the existence of a differentiable function $\mathrm{g}(\pi_T^i)$ such that we have $\mathcal{G}(\pi_T^i, g(\pi_T^i)) = 0$ locally in a small neighbourhood around (π_T^i, φ^*). In view of this fact and Assumption 5.2 it then follows that we can write $D_1 C(\pi_T^i + \theta^i \varphi^*) = \mathbb{K}(\pi_T^i)$, where $\mathbb{K}$ is a function which is locally constant in a small neighbourhood around π_T^i such that we have

$$u^i\left(f^i\left(-\varphi^*, \psi_T^i\right) + \lambda C \circ f^i\left(-\varphi^*, \pi_T^i\right)\right) = u^i\left(\psi_T^i + \frac{\left(\pi_T^i\right)^2}{2\theta^i} + \lambda \mathbb{K}\left(\pi_T^i\right) \pi_T^i - \frac{\theta^i \lambda^2}{2}\mathbb{K}^2\left(\pi_T^i\right) + \lambda C\left(\theta^i \lambda \mathbb{K}\left(\pi_T^i\right)\right)\right)$$

In order to show the non–degenracy of $F^{i,\lambda}$ in the manipulable case, we again consider the zero strategy $\mathcal{V}^{i,0}$ and note that $\psi_T^{i,0} = \psi_\kappa^{i,0}$. Moreover, it is immediate from the definition of auxiliary state variables that π_T^i is not affected by the strategy of investor i, which gives us

$$\begin{aligned}
&F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) \\
&\quad \geq \mathbb{E}\left[u^i\left(\psi_\kappa^i + \frac{\left(\pi_T^i\right)^2}{2\theta^i} + \lambda \mathbb{K}\left(\pi_T^i\right) \pi_T^i - \frac{\theta^i \lambda^2}{2}\mathbb{K}^2\left(\pi_T^i\right) + \lambda C\left(\theta^i \lambda \mathbb{K}\left(\pi_T^i\right)\right)\right)\right] \\
&\quad \geq \mathbb{E}\left[u^i\left(\psi_\kappa^i + \lambda \mathbb{K}\left(\pi_T^i\right) \pi_T^i - \frac{\theta^i \lambda^2}{2}\mathbb{K}^2\left(\pi_T^i\right) + \lambda C\left(\theta^i \lambda \mathbb{K}\left(\pi_T^i\right)\right)\right)\right]
\end{aligned}$$

In view of the equation above and recalling the exponential form of the utility function, we invoke Cauchy–Schwarz inequality and note that in order to establish the non–degenracy of $F^{i,\lambda}$ it suffices to show that given some positive constant K we have

$$K\mathbb{E}\left[\exp\left(K\mathbb{K}\left(\pi_T^i\right)\pi_T^i\right)\right]\mathbb{E}\left[\exp\left(K\mathbb{K}^2\left(\pi_T^i\right)\right)\right]\mathbb{E}\left[\exp\left(KC\left(\theta^i\lambda\mathbb{K}\left(\pi_T^i\right)\right)\right)\right] < \infty$$

It is immediate in view of Assumption 5.1 that the third expectation on the right–hand side of the equation above is finite, while in addition Assumption 5.2 also implies that the function $\mathbb{K}(\pi_T^i)$ is bounded from which it is straightforward to deduce that the second expectation on the right–hand side of the equation above is finite. Moreover, it also implies that there exists a positive constant $\hat{K}$ such that we can bound the first expectation above as follows

$$\mathbb{E}\left[\exp\left(K\mathbb{K}\left(\pi_T^i\right)\pi_T^i\right)\right] \leq \mathbb{E}\left[\exp\left(\hat{K}\pi_T^i\right)\right]$$

We once again recall the exponential form of the utility function in conjunction with the definition of the auxiliary state variable π_T^i along with CauchySchwarz inequality to assert the existence of a positive constant K such that we have

$$\mathbb{E}\left[\exp\left(\hat{K}\pi_T^i\right)\right] \leq K\mathbb{E}\left[\exp\left(K\int_\kappa^T \left|x_s^{-i}\right| ds\right)\right]\mathbb{E}\left[\exp\left(K(B_T - B_\kappa)\right)\right]$$

The first term on the right–hand side of the inequality above is finite in view of Definition 3.2. Further, the second term on the right–hand side of the inequality above is bounded above by $\exp(K(T-\kappa)/2)$ and is finite, thus implying the non–dengenracy of $F^{i,\lambda}$. Next, given $\epsilon > 0$, we consider an ϵ–optimal admissible strategy $\mathcal{V}^{i,\epsilon}$ for the auxiliary hedging problem in the manipulable case, whose existence is implied by the definition of $F^{i,\lambda}$. In view of the preceding arguments, in the manipulable case the ϵ–optimal admissible strategy $\mathcal{V}^{i,\epsilon}$ must satisfy

$$\begin{aligned}
&F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \\
&\quad \leq \mathbb{E}\left[u^i\left(\psi_T^{i,\epsilon} + \frac{\left(\pi_T^{i,\epsilon}\right)^2}{2\theta^i} + \lambda \mathbb{K}\left(\pi_T^{i,\epsilon}\right) \pi_T^{i,\epsilon} - \frac{\theta^i \lambda^2}{2}\mathbb{K}^2\left(\pi_T^{i,\epsilon}\right) + \lambda C\left(\theta^i \lambda \mathbb{K}\left(\pi_T^{i,\epsilon}\right)\right)\right)\right]
\end{aligned} \tag{A.13}$$

Note that $\psi_T^{i,\epsilon}$ and $\pi_T^{i,\epsilon}$ on the right–hand side of the equation above denote the terminal value of the corresponding auxiliary state variables when investor i and investor $-i$ employ strategies $\mathcal{V}^{i,\epsilon}$ and X^{-i} respectively. Next, from Lemma 5.4, we can dedue the existence of a sequence of admissible strategies $\left\{\mathcal{V}^{i,\epsilon_n} \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)\right\}$ such that $\rho\left(\mathcal{V}^{i,\epsilon_n}, \mathcal{V}^{i,\epsilon}\right) \to 0$, as $n \to \infty$. Also, from Lemma 5.5, we know that $Z_T^{i,\epsilon_n} \to Z_T^{i,\epsilon}$, $\mathbb{P}$-almost surely as $n \to \infty$ (at least along a subsequence), where Z^{i,ϵ_n} denotes the controlled auxiliary state process when investor i and investor $-i$ employ strategies $\mathcal{V}^{i,\epsilon_n}$ and X^{-i} respectively. Thus, in a similar vein as the non–manipulable case, to prove the desired claim we need to show that

$$\lim_{n\to\infty} \mathbb{E}\left[u^i\left(f^i\left(-\varphi_n^*, \psi_T^{i,\epsilon_n}\right) + \lambda C \circ f^i\left(-\varphi_n^*, \pi_T^{i,\epsilon_n}\right)\right)\right] = \mathbb{E}\left[u^i\left(f^i\left(-\varphi^*, \psi_T^{i,\epsilon}\right) + \lambda C \circ f^i\left(-\varphi^*, \pi_T^{i,\epsilon}\right)\right)\right]$$

In view of preceding arguments, we know that the right hand of the equation above is finite. Further, we know that the auxiliary state variable π_T^i is not dependent on the strategy of investor i by definition, which in conjunction with (A.12) implies that $\pi_T^{i,\epsilon} = \pi_T^{i,\epsilon_n} = \pi_T^i$ and that $\varphi^* = \varphi_n^*$ for all $n \in \mathbb{N}$. Then, in view of the continuity of the utility function u^i and the flow f^i it follows that we have $\mathbb{P}$–almost surely

$$u^i\left(f^i\left(-\varphi_n^*, \psi_T^{i,\epsilon_n}\right) + \lambda C \circ f^i\left(-\varphi_n^*, \pi_T^{i,\epsilon_n}\right)\right) \longrightarrow u^i\left(f^i\left(-\varphi^*, \psi_T^{i,\epsilon}\right) + \lambda C \circ f^i\left(-\varphi^*, \pi_T^{i}\right)\right) \text{ as } n \to \infty$$

Thus, in view of the above, in order to show the desired convergence of the sequence of expected utilities associated with piecewise constant strategies to expected utility from the ϵ–optimal control, it remains to show that the sequence of expected utilities corresponding to piecewise constant strategies is uniformly integrable, that is, we need to show

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} u^i\left(\psi_T^{i,\epsilon_n} + \frac{\left(\pi_T^i\right)^2}{2\theta^i} + \lambda\mathbb{K}\left(\pi_T^i\right)\pi_T^i - \frac{\theta^i\lambda^2}{2}\mathbb{K}^2\left(\pi_T^i\right) + \lambda C\left(\theta^i\lambda\mathbb{K}\left(\pi_T^i\right)\right)\right)\right] < \infty$$

In view of the exponential form of the utility function and through an iterated application of Cauchy–Schwarz inequality, we observe that in order to show that the inequality above holds it suffices to show that given some positive constant K we have

$$\begin{aligned} &K\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\psi_T^{i,\epsilon_n}\right)\right] \times \mathbb{E}\left[\exp\left(-\frac{\gamma^i}{\theta^i}K\left(\pi_T^i\right)^2\right)\right] \times \mathbb{E}\left[\exp\left(K\mathbb{K}\left(\pi_T^i\right)\pi_T^i\right)\right] \\ &\times \mathbb{E}\left[\exp\left(K\mathbb{K}^2\left(\pi_T^i\right)\right)\right] \times \mathbb{E}\left[\exp\left(KC\left(\theta^i\lambda\mathbb{K}\left(\pi_T^i\right)\right)\right)\right] < \infty \end{aligned}$$

Note that in view of the preceding arguments, we know that the last three expectation on the left–hand side of the inequality above are finite. It then remains to show that the first two expectations are finite. To this end, note that by definition of the auxiliary state variable ψ_T^i, the exponential form of the utility function and Cauchy–Schwarz inequality we can ascertain the existence of a positive constant $\hat{K}$ such that we have

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\psi_T^{i,\epsilon_n}\right)\right] \leq \hat{K}\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(\hat{K}\int_\kappa^T \vartheta_s^{i,\epsilon_n} x_s^{-i}\, ds\right)\right] \mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(\hat{K}\int_\kappa^T \vartheta_s^{i,\epsilon_n} dB_s\right)\right]$$

Observe that by Hölder's inequality, we obtain the following upper bound for the first term on the right–hand side of the equation above

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_\kappa^T \vartheta_s^{i,\epsilon_n} x_s^{-i}\, ds\right)\right] \leq \mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\left(\sup_{s\in[\kappa,T]}\left|\vartheta_s^{i,\epsilon_n}\right|\right)\int_\kappa^T \left|x_s^{-i}\right| ds\right)\right]$$

It is straightforward to check that in view of Definition 3.2 and Definition 4.1 the right–hand side of the inequality above is finite. In a similar vein, we appeal to (Protter, 2004, Theorem 43, Page 140) in conjunction with Definition 4.1 in order to ascertain the existence of a positive constant K such that we have

$$\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_{\kappa}^{T} \vartheta_s^{i,\epsilon_n} dB_s\right)\right] \leq K\mathbb{E}\left[\sup_{n\in\mathbb{N}} \exp\left(K\int_{\kappa}^{T} \left(\vartheta_s^{i,\epsilon_n}\right)^2 ds\right)\right] < \infty$$

Next, we note that by definition of the auxiliary state variable π_T^i, the exponential form of the utility function and Cauchy–Schwarz inequality we can ascertain the existence of a positive constant $\hat{K}$ such that we have

$$\mathbb{E}\left[\exp\left(-\frac{\gamma^i}{\theta^i} K\left(\pi_T^i\right)^2\right)\right] \leq \hat{K}\mathbb{E}\left[\exp\left(\hat{K}\int_{\kappa}^{T} x_s^{-i}\, ds\right)^2\right] \mathbb{E}\left[\exp\left(-\frac{\gamma^i}{\theta^i} \hat{K}(B_T - B_\kappa)^2\right)\right]$$

In view of Definition 3.2 it is immediate that first term on the right–hand side of the inequality above is finite. Likewise, we appeal to (Mansuy and Yor, 2008, Result 2.1.1, Page 18) with drift coefficient normalised to zero (in the limit) in order to ascertain that the second expectation on the right–hand side of the inequality above is finite. In view of the preceding arguments and (A.13), we thus arrive at the following

$$\begin{aligned} F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon &\leq \mathbb{E}\left[u^i\left(f^i\left(-\varphi^*, \psi_T^{i,\epsilon}\right) + \lambda C \circ f^i\left(-\varphi^*, \pi_T^{i,\epsilon}\right)\right)\right] \\ &= \limsup_{n\to\infty} \mathbb{E}\left[u^i\left(f^i\left(-\varphi_n^*, \psi_T^{i,\epsilon_n}\right) + \lambda C \circ f^i\left(-\varphi_n^*, \pi_T^{i,\epsilon_n}\right)\right)\right] \end{aligned}$$

Moreover, as an immediate consequence of the inequality above we ascertain that

$$\begin{aligned} &F^{i,\lambda}\left(\kappa, \left[\pi_\kappa^i, \mu_\kappa^i, \mu_\kappa^{-i}, \psi_\kappa^i, \psi_\kappa^{-i}\right]^{\mathrm{T}}; X^{-i}\right) - \epsilon \\ &\quad\leq \limsup_{n\to\infty} \sup_{\mathcal{V}^{i,n} \in \mathcal{A}_\kappa^{\mathrm{m},a,pc}(T_n)} \mathbb{E}\left[u^i\left(f^i\left(-\varphi_n^*, \psi_T^{i,\epsilon_n}\right) + \lambda C \circ f^i\left(-\varphi_n^*, \pi_T^{i,\epsilon_n}\right)\right)\right] \end{aligned}$$

The desired claim then follows by combining the inequality above with (A.10) given that the choice of ϵ was arbitrary.

□